

Fixed-Points of Social Choice: An Axiomatic Approach to Network Communities

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Abstract

We provide the first social choice theory approach to the question of what constitutes a community in a social network. Inspired by social choice theory in voting and other contexts [2], we start from an abstract social network framework, called *preference networks* [3]; these consist of a finite set of members and a vector giving a total ranking of the members in the set for each of them (representing the preferences of that member).

Within this framework, we axiomatically study the formation and structures of communities. Our study naturally involves two complementary approaches. In the first, we apply social choice theory and define communities indirectly by postulating that they are fixed points of a preference aggregation function obeying certain desirable axioms. In the second, we directly postulate desirable axioms for communities without reference to preference aggregation, leading to a natural set of eight community axioms.

These two approaches allow us to formulate and analyze community rules. We prove a taxonomy theorem that provides a *structural characterization* of the family of those community rules that satisfies all eight axioms. The structure is actually quite beautiful: the family satisfying all eight axioms forms a bounded lattice under the natural intersection and union operations of community rules. The taxonomy theorem also gives an explicit characterization of the most comprehensive community rule and the most selective community rule consistent with all community axioms. This structural theorem is complemented with a *complexity result*: we show that while identifying a community by the selective rule is straightforward, deciding if a subset satisfies the comprehensive rule is coNP-complete. Our studies also shed light on the limitations of defining community rules solely based on preference aggregation. In particular, we show that many aggregation functions lead to communities which violate at least one of our community axioms. These include any aggregation function satisfying Arrow's independence of irrelevant alternative axiom as well as commonly used aggregation schemes like the Borda count or generalizations thereof. Finally, we give a polynomial-time rule consistent with seven axioms and weakly satisfying the eighth axiom.

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1 Introduction: Formulating Preferences and Communities

A fundamental problem in network analysis is the characterization and identification of subsets of nodes in a network that have significant structural coherence. This problem is usually studied in the context of *community identification* and *network clustering*. Like other inverse problems in machine learning, this one is conceptually challenging: There are many possible ways to measure the degree of coherence of a subset and many possible interpretations of affinities to model network data. As a result, various seemingly reasonable/desirable conditions to qualify a subset as a community have been studied in the literature [15, 21, 18, 11, 3, 10, 8, 17, 22, 6]. The fact that there are an exponential number of candidate subsets to consider makes direct comparison of different community characterizations quite difficult.

Among the challenges in the study of communities in a social and information network are the following two basic mathematical problems:

- **Extension of individual affinities/preferences to community coherence:** A (social) network usually represents pairwise interactions among its members, while the notion of communities is defined over its larger subsets. Thus, to model the formation of communities, we need a set of consistent rules to extend the pairwise relations or individual preferences to community coherence.
- **Inference of missing links:** Since networks typically are sparse, we also need methods to properly infer the missing links from the given network data.

In this paper, we take what we believe is a novel and principled approach to the problem of community identification. Inspired by the classic work in social choice theory [2], we propose an axiomatic approach towards understanding network communities, providing both a framework for comparison of different community characterizations, and relating community identification to well-studied problems in social choice theory [2]. Here, we focus on the problem of defining community rules and coherence measures from individual preferences presented in the input social/information network, but we think that this study will also provide the foundation for an axiomatic approach to the problem of inferring missing links. We plan to address this second problem in a subsequent paper, which will use this paper as a foundation.

Through the lens of axiomatization, we examine both mathematical and complexity-theoretic structures of communities that satisfy a community rule or a set of community axioms. We also study the stability of network communities, and design algorithms for identifying and enumerating communities with desirable properties.

While the study initiated here is conceptual, we believe it will ultimately enable a more principled way to choose among community formation models for interpretation of current experiments, and also suggest future experiments.

1.1 Preference Networks

Before presenting the highlights of our work, we first define an *abstract* social network framework which enables us to focus on the axiomatic study of community rules. This framework is inspired by social choice theory [2] and was first used in [3] in the context of community identification for modeling social networks with complete preference information. We will refer to each instance of this framework as a *preference network*. Below, for a non-empty finite set V , let $L(V)$ denote

the set of all linear orders on V , represented, e.g., by the set of all bijections $\pi : V \rightarrow [1 : |V|]$, where as usual, $[n : m]$ is the set $\{n, n+1, \dots, m\}$. Alternatively, π can be represented by the ordered list $\pi = [x_1, x_2, \dots, x_{|V|}]$, where $x_i \in V$ is such that $\pi(x_i) = i$; in our notation, $\pi(x)$ thus represents the rank of x in the ordered list $\pi = [x_1, x_2, \dots, x_{|V|}]$.

Definition 1 (Preference Networks). A preference network is a pair $A = (V, \Pi)$, where V is a non-empty finite set and Π is a preference profile on V , defined as an element $\Pi = \{\pi_i\}_{i \in V} \in L(V)^V$. Here π_i specifies the total ranking¹ of V in the order of i 's preference: $\forall s, u, v \in V$, s prefers u to v , denoted by $u \succ_{\pi_s} v$, if and only if $\pi_s(u) < \pi_s(v)$.

As argued in [3], a real-life social network may be viewed as sparse, observed social interactions of an underlying latent preference network. In this view, the communities of a preference network may be considered to be the ground truth set of potential communities in its observed social network.

1.2 Highlights of the Paper

Our main contribution is an axiomatic framework for studying community formation in preference networks, and mathematical, complexity-theoretic, and algorithmic investigation of community structures in this framework. Our work on axiomatization of network communities can be organized into two related parts: (1) communities as fixed points of social choice aggregation functions; and (2) communities via direct axiomatic characterization. In the second part, we specify eight axioms we would like the communities to obey, and find conditions under which such communities exist. In the first part, we specify social choice aggregation functions for which the communities will be fixed points; this first method allows for an “indirect” axiomatic characterization in that the aggregation functions themselves could be taken to obey axioms [2, 24], which would then indirectly characterize the communities which result as fixed points.

Communities as fixed points of social choice

Our approach of starting from preference networks to study communities naturally connects community formation to social choice theory [2], which provides a theoretical framework for understanding the problem of combining individual preferences into a collective preference or decision. In this first part of our analysis, we use *preference aggregation functions* studied in social choice theory [2] to characterize communities by defining communities as fixed points of a preference aggregation function.

Since real-world voting schemes and preference aggregation functions do not always produce a total order, we will use the following notation in the definition below. Let $\overline{L(V)}$ denote the set of all *ordered partitions* of V . For a $\sigma \in \overline{L(V)}$, for $i, j \in V$, we use $i \succ_{\sigma} j$ to denote that i is *strictly preferred to* j (that is, i and j belong to different partitions, and the partition containing

¹ In broader settings, one may want to consider preferences that allow *indifference* or partially ordered preferences, or both. One may also model a social network by a cardinal affinity network that specify each member's preference by a weighted affinity vector, for example with weights from $[0, 1]$ where 1 and 0, respectively, represent the highest and lowest preferences. To distinguish ordinal and cardinal preferences, we refer to the latter as an *affinity network*. Both models are referred to as *affinity systems* in [3]. For simplicity of exposition, we first focus on preference networks. In Section 7, we discuss the possible extension of our framework.

i is ahead of the partition containing j in σ). In this case, we also say $j \prec_\sigma i$. We will use $i \succeq_\sigma j$ to denote that $i \succ_\sigma j$ or i and j are in the same partition.

To continue, we need some notions motivated by social choice theory. In this context, V will be considered a set of “candidates”. We’ll also need a set of possible voters, \mathcal{S} , which is assumed to be a countable set – if not otherwise specified, we identify \mathcal{S} with the positive integers \mathbb{N} . With a slight abuse of notation, we denote the union of $L(V)^S$ over all non-empty finite $S \subseteq \mathcal{S}$ by $L(V)^*$. A *preference aggregation function* is then defined to be an arbitrary function $F : L(V)^* \rightarrow \overline{L(V)}$. Given a non-empty finite set of voters S and a preference profile $\Pi_S = \{\pi_s : s \in S\} \in L(V)^S$, the image $F(\Pi_S)$ is called the *aggregated preference*² of S .

Definition 2. (COMMUNITIES AS FIXED POINTS OF SOCIAL CHOICE) *Let $A = (V, \Pi)$ be a preference network, $F : L(V)^* \rightarrow \overline{L(V)}$ be a preference aggregation function, and $\emptyset \neq S \subseteq V$. S is called a community of A with respect to F if and only if $u \succ_{F(\Pi_S)} v$, $\forall u \in S, v \in V - S$.*

The function \mathcal{C}_F mapping A into the set of communities defined above is called the fixed point rule with respect to F . If F is not specified, i.e., if there exists an F such that $\mathcal{C} = \mathcal{C}_F$, we call \mathcal{C} simply a fixed point rule.

Informally, this definition says that a community is a subset $S \subseteq V$ such that, when we aggregate the preferences of all its members, the resulting aggregated preference puts the members S as the top $|S|$ elements.³ In other words, under the aggregation function F , the members of the community “vote” for themselves. Thus, S is a *fixed point* of its aggregated preference. The community characterization of Definition 2 generalizes the following concept of self-determination of [3]:

Definition 3. (B³CT COMMUNITIES) *Let $A = (V, \Pi)$ be a preference network. For $\emptyset \neq S \subseteq V$ and $i \in V$, let $\phi_S^\Pi(i)$ denote the number of votes that member i would receive if each member $s \in S$ was casting one vote for each of its $|S|$ most preferred members according to its preference π_s . In other words, $\phi_S^\Pi(i) = |\{s : (s \in S) \ \& \ (\pi_s(i) \in [1 : |S|])\}|$. Then, S is B³CT-self-determined if everyone in S receives more votes from S than everyone outside S .*

It is easy to see that the B³CT voting rule is an instance of a fixed-point rule, with preference aggregation function F defined by $v \succ_{F(\Pi_S)} w$ iff $\phi_S^\Pi(v) > \phi_S^\Pi(w)$.

We will also refer to a community according to Definition 2 as an *F-self-determined community*. We are particularly interested in those aggregation functions that satisfy various axioms in social choice theory [2], since this enables us to utilize established social choice theory to study all conceivable self-determination community rules within one unified framework. For example, it allows us to reduce the fairness analysis for community formation to the fairness of preference aggregation functions.

Arrow’s celebrated impossibility theorem and subsequent work in social choice theory [2] point to both challenges and exciting opportunities for understanding communities in preference networks. Recall that Arrow’s theorem states that for $n > 2$, no (strictly linear) preference aggregation function satisfies all of the following three axiomatic conditions: Unanimity, Independence of Irrelevant Alternatives, and Non-Dictatorship (see Section 2 for definitions.) On the

²Note that in our notation without further requiring F to satisfy additional conditions, the labels in S matter: e.g., even if π_1 and π_2 are the same permutation of $[n]$, the values of $F(\Pi_{\{1\}})$ and $F(\Pi_{\{2\}})$ can be different.

³In the case of ties, we allow for ties among the top $|S|$ members, as well as among the lower ranked members, but not between the top $|S|$ members and anyone below.

other hand, preference aggregation functions exist if one relaxes any of these conditions. For instance, the well-known Borda count [23] is a unanimous voting method with no dictators.

In this paper, we will examine the impact of preference aggregation functions on the structure of the self-determined communities that they define, as well as the limitations of formulating community rules solely based on preference aggregation. See below for more discussion.

Communities via direct axiomatic characterization

In this second approach, we will use a more direct axiomatic characterization to study network communities. To this end, we use a set-theoretical *community function* as a means to characterize a community rule.

Definition 4 (Community Functions). *Let \mathcal{A} denote the set of all preference networks. A community function is a function \mathcal{C} that maps a preference network $A = (V, \Pi)$ to a characteristic function of non-empty subsets of V . In other words, $\mathcal{C}(A) \subseteq 2^{2^V - \{\emptyset\}}$ is an indicator function of $2^V - \{\emptyset\}$. We say a subset $S \subseteq V$ is a community in a preference network $A = (V, \Pi)$ according to a community function \mathcal{C} if and only if $S \in \mathcal{C}(A)$. To simplify our notation, for $A = (V, \Pi)$ we often write $\mathcal{C}(V, \Pi)$ instead of $\mathcal{C}((V, \Pi))$.*

We use axioms to state properties, such as fairness and consistency, that a desirable community function should have when applied to all preference networks. An example is the property that the community function should be isomorphism-invariant: Here an *isomorphism* between two preference networks $A = (V, \Pi)$ and $A' = (V', \Pi')$ is a bijection $\sigma : V \rightarrow V'$ such that $\Pi' = \sigma(\Pi)$, i.e., such that for all $s, v \in V$, $\pi'_{\sigma(s)}(\sigma(v)) = \pi_s(v)$, and two preference networks A and A' are *isomorphic* to each other if there exists such an isomorphism. Isomorphism invariance then requires that for any pair of isomorphic preference networks $A = (V, \Pi)$ and $A' = (V', \Pi')$ and any isomorphism σ between A and A' , if $S \subset V$ is a community in A , then $\sigma(S)$ should still be a community in the A' . Another example is the property of *monotonic characterization*: If S is a community in $A = (V, \Pi)$, then S should remain a community in every preference network $A' = (V, \Pi')$ such that for all $u, s \in S$ and $v \in V$, if $u \succ_{\pi_s} v$ then $u \succ_{\pi'_s} v$.

In Section 2, we propose a natural set of eight desirable community axioms. Six of them, including both examples above, provide a positive characterization of communities. These axioms concern the consistency, fairness, and robustness of a community function, as well as the community structures when a preference network is embedded in a larger preference network. The other two axioms address the necessary stability and self-approval conditions that a community should satisfy.

Constructing and Analyzing Community Rules

While Definition 4 is convenient for the study of the mathematical structure of our theory, community identification is a computational problem as much as a mathematical problem. Thus, it is desirable that communities can be characterized by a constructive community function \mathcal{C} that is:

- **Consistent:** \mathcal{C} satisfies all (or nearly all) axioms;
- **Constructive:** Given a preference network $A = (V, \Pi)$, and a subset $S \subseteq V$, one can determine in polynomial-time (in $n = |V|$) if $S \in \mathcal{C}(A)$.

- **Samplable:** One can efficiently obtain a random sample of $\mathcal{C}(A)$.
- **Enumerable:** One can efficiently enumerate $\mathcal{C}(A)$, for instance, in time $O(n^k \cdot |\mathcal{C}(A)|)$ for a constant k .

Our two axiomatic approaches allow us to formulate a rich family of community rules and analyze their properties. Using the fixed-point rule, we can define a constructive community function based on any polynomial-time computable aggregation function. Alternatively, we can use one axiom or a set of axioms as a community rule. We can also define a community rule by the intersection of a fixed-point rule and a set of axioms. In this paper, we aim to *characterize the community rules that satisfy a set of “reasonable” axioms*, and address the basic questions:

- *Is there an aggregation function leading to a community rule satisfying this set of “reasonable” axioms?*
- *What is the complexity of the community rules based on these axioms?*
- *How are different community rules satisfying our axioms related to each other? For example, given two community rules \mathcal{C}_1 and \mathcal{C}_2 satisfying our axioms, does the rule \mathcal{C} defined by $\mathcal{C}(A) := \mathcal{C}_1(A) \cap \mathcal{C}_2(A)$ obey our axioms as well?*

Structural and Complexity-Theoretic Results

Our main structural result is a taxonomy theorem that provides a complete characterization of the most comprehensive community rule and the most selective community rule consistent with all our community axioms. This result illustrates an interesting contrast to the classic axiomatization result of Arrow [2] and the more recent result of Kleinberg on clustering [9] that inspired our work. Unlike voting or clustering where the basic axioms lead to impossibility theorems, the preference network framework offers a natural community rule, which we call the *Clique Rule*, that is intuitively fair, consistent, and stable, although selective (See Section 4 for more details): S is a community according the Clique Rule iff each member of S prefers every member of S over every non-member. Indeed the Clique Rule satisfies all our axioms. Our analysis then leads us to a community rule which is consistent with all axioms – we call it the *Comprehensive Rule* – such that for any community rule \mathcal{C} satisfying all axioms and all preference network A , $\mathcal{C}_{\text{clique}}(A) \subseteq \mathcal{C}(A) \subseteq \mathcal{C}_{\text{comprehensive}}(A)$. Perhaps more interesting, under the natural operations of union and intersections, the set of all community rules satisfying all our axioms becomes a lattice with $\mathcal{C}_{\text{clique}}(A)$ and $\mathcal{C}_{\text{comprehensive}}(A)$ forming a lower and upper bound, respectively.

We complement this structural theorem with a complexity result: we show that while identifying a community by the Clique Rule is straightforward, it is **coNP-complete** to determine if a subset satisfies the comprehensive rule.

Our studies also shed light on the limitations of formulating community rules solely based on preference aggregation. In particular, we show that many aggregation functions lead to communities which violate at least one of our community axioms. We give two impossibility-like theorems.

1. Any fixed-point rule based on commonly used aggregation schemes like Borda count or generalizations thereof – such as the B³CT self-determination rule – is inconsistent with (at least) one of our axioms.

2. For any aggregation function satisfying Arrow’s independence of irrelevant alternative axiom, its fixed-point rule must violate one of our axioms.

Finally, using our direct axiomatic framework, we analyze the following natural constructive community function inspired by preference aggregations.

Definition 5 (Harmonious Communities). *A non-empty subset $S \subseteq V$ is a harmonious community of a preference network $A = (V, \Pi)$ if for all $u \in S$ and $v \in V - S$, the majority of $\{\pi_s : s \in S\}$ prefer u over v .*

We will show that the harmonious community rule is consistent with seven axioms and satisfies a weaker form of the eighth axiom. In addition, various stable versions of harmonious communities (see the discussion below) enjoy some degree of samplability and enumerability.

Stability of Communities and Algorithms

In real-world social interactions, some communities are more stable or durable than others when people’s interests and preferences evolve over time. For example, some music bands stay together longer than others. Inspired by the work of [3] and Mishra *et al.* [11] on modeling this phenomenon, we examine the impact of stability on the community structure.

To motivate our discussion, we first recall the main definition and result of [3]:

Definition 6. *For $0 \leq \beta < \alpha \leq 1$, a non-empty subset $S \subseteq V$ is an (α, β) -B³CT community in $A = (V, \Pi)$ iff $\phi_S^\Pi(u) \geq \alpha \cdot |S| \ \forall u \in S$ and $\phi_S^\Pi(v) < \beta \cdot |S| \ \forall v \notin S$.*

It was shown in [3] that, in any preference network, there are only polynomially many stable B³CT communities when the parameters α, β are constants, and they can be enumerated in polynomial time, showing that the strength of community coherence has both structural and computational implications.

In Section 6, we consider several stability conditions in our axiomatic community framework. In one direction, we examine the structure of the communities (defined by a fixed-point community rule) that remain self-determined even after a certain degree of perturbation in its members’ preferences. In this context, for example, we can reinterpret the B³CT-stability defined above as follows: A subset $S \subseteq V$ is an (α, β) -B³CT community in a preference network A if it remains self-determined when $|S| \cdot (\alpha - \beta)/2$ members of S make arbitrary changes to their preferences. In the other direction, we consider some notions of stability derived directly from the social-choice based community framework where members of a community separate themselves from the rest. We can further use the separability as a measure of the community strength and stability to capture the intuition that stronger communities are also themselves more integrated. As a concrete example, we show in Section 6 that there are a quasi-polynomial number of stable harmonious communities for all these notions of stability. This result demonstrates that there exists a constructive community function that essentially satisfies all our axioms, whose stable communities are quasi-polynomial-time samplable and enumerable.

2 Coherent Communities: Axioms

In this section, we define our eight core axioms, give a more formal treatment of social choice axioms, and examine several properties of community rules and the relations these have with each other.

2.1 Lexicographic Preference

The following notion will be crucial in several parts of this paper, and is implicitly used in our first two axioms below.

Definition 7 (Lexicographical Preferences). *Given a preference network (V, Π) and two non-empty disjoint subsets G and G' of equal size, we say that $s \in V$ lexicographically prefers G' over G if there exists a bijection $f_s : G \rightarrow G'$ such that $f_s(u) \succ_{\pi_s} u$ for all $u \in G$.*

We say that a group $T \subset V$ lexicographically prefers G' over G if every member $s \in T$ lexicographically prefers G' to G , i.e., if there exists a set of bijections $\{f_s : G \rightarrow G' \mid s \in T\}$ such that $f_s(u) \succ_{\pi_s} u$ for all $u \in G$ and all $s \in T$.

Note that, in contrast to the standard lexicographical order, lexicographical preference is only a partial order. The notion is motivated by the following proposition.

Proposition 1. *Let $\pi \in L(V)$, let G and G' be disjoint subsets of V with $|G| = |G'|$. Let $G[i]$ (and $G'[i]$) be the i^{th} highest ranked element of G (and G') according to π . Then there exists a bijection $f : G \rightarrow G'$ such that for all $g \in G$, $f(g) \succ_{\pi} g$ if and only for all $i \in [1 : |G|]$, $G'[i] \succ_{\pi} G[i]$.*

Proof. Suppose f satisfies the condition of the proposition. Then $G'[1] \succeq_{\pi} f(G[1]) \succ_{\pi} G[1]$. If $G'[1] \neq f(G[1])$, define h to be the bijection on G' which exchanges $G'[1]$ and $f(G[1])$, and define $\tilde{f} = g \circ f$. Then $G'[1] = \tilde{f}(G[1]) \succ_{\pi} G[1]$ while \tilde{f} still satisfies the condition of the proposition. Removing $G[1]$ from G and $G'[1]$ from G' , we continue by induction to prove the only if statement. The if statement is obvious - just define f by $f(G[i]) = G'[i]$. \square

2.2 Axioms for Community Functions

For the following definitions, fix a ground set V and a community function \mathcal{C} .

Axiom 1 (Group Stability (GS)). *If Π is a preference profile over V and $S \in \mathcal{C}(V, \Pi)$, then S is GROUP STABLE with respect to Π . Here a subset $S \subset V$ is called group stable with respect to Π if for all non-empty $G \subsetneq S$, all $G' \subset V - S$ of the same size as G , and all tuples of bijections, $(f_i : G \rightarrow G', i \in S - G)$, there exists $s \in S - G$, $u \in G$ such that $u \succ_{\pi_s} f_s(u)$.*

This axiom provides a type of game-theoretic stability [14, 13, 4, 19, 20], and states that no subgroup in a community can be replaced by an equal-size group of non-members that are lexicographically preferred by the remainder of the community members. For instance, if the subgroup is of size 1, this means that there is no outsider that is universally preferred to this member, excluding that member's own opinion. On the other end of the spectrum, if the subgroup is all but one person, then group stability states that there must be someone from that member's top choices, and thus represents a type of individual rationality condition. Note that the set V is vacuously group stable for all Π .

Axiom 2 (Self-Approval (SA)). *If Π is a preference profile over V , and $S \in \mathcal{C}(V, \Pi)$ then S is SELF-APPROVING with respect to Π . Here a subset $S \subset V$ is called self-approving with respect to Π if for all $G' \subseteq V - S$ of the same size as S , and all tuples of bijections $(f_i : S \rightarrow G', i \in S)$ there exists $s, u \in S$, such that $u \succ_{\pi_s} f_s(u)$.*

Axiom SA uses the same partial ordering of groups as the first, and requires that there is no outside group of the same size as S which is lexicographically preferred to S by everyone in S . It generalizes the intuition that a singleton should be a community only if that member prefers herself to everyone else. Note that any set S of size larger than $|V|/2$ is vacuously self-approving for all Π .

Axiom 3 (Anonymity (A)). *Let $S, S' \subset V$ and Π, Π' be such $S' = \sigma(S)$ and $\Pi' = \sigma(\Pi)$ for some permutation $\sigma : V \rightarrow V$. Then $S \in \mathcal{C}(V, \Pi) \iff S' \in \mathcal{C}(V, \Pi')$.*

A staple axiom, Anonymity, states that labels should have no effect on a community function.

Axiom 4 (Monotonicity (Mon)). *Let $S \subset V$. If Π and Π' are such that for all $s \in S$*

$$u \succ_{\pi'_s} v \implies u \succ_{\pi_s} v \quad \text{for all } u \in S, v \in V$$

then $S \in \mathcal{C}(V, \Pi') \implies S \in \mathcal{C}(V, \Pi)$.

The Axiom Monotonicity states that, if a member of a community gets promoted without negatively impacting other members, then that subset must remain a community. Thus this axiom reflects the fact that high positions imply greater affinities towards those people. Note that Mon also allows non-members to change arbitrarily, as long as their positions relative to any members remains the same or worse.

Axiom 5 (Coherence Robustness of Non-Members (CRNM)). *Let $S \subset V$. If Π and Π' are such that for all $s, t \in S$*

$$v \succ_{\pi'_s} w \iff v \succ_{\pi'_t} w \quad \text{for all } v, w \notin S$$

and

$$\pi'_s(u) = \pi_s(u) \quad \text{for all } u \in S,$$

then $S \in \mathcal{C}(V, \Pi') \implies S \in \mathcal{C}(V, \Pi)$.

Axiom 6 (Coherence Robustness of Members (CRM)). *Let $S \subset V$. If Π and Π' are such that for all $s, t \in S$ we have*

$$u \succ_{\pi'_s} w \iff u \succ_{\pi'_t} w \quad \text{for all } u, w \in S$$

and

$$\pi'_s(v) = \pi_s(v) \quad \text{for all } v \notin S,$$

then $S \in \mathcal{C}(V, \Pi') \implies S \in \mathcal{C}(V, \Pi)$.

The two Coherence Robustness Axioms reflect the fact that, if community members agree about their preferences concerning either members or non-members, they are less likely to be a community. In the case of non-members, agreement implies that some non-member is more preferred and therefore more likely to break up the community. Contrariwise, in the case of members, agreement implies some member is less preferred and more likely to be ousted.

Axiom 7 (World Community (WC)). *For all preference profiles Π , $V \in \mathcal{C}(V, \Pi)$.*

To state the next axiom, we define the projection $A|_{V'}$ of a preference network $A = (V, \Pi)$ onto a subset $V' \subset V$ as the preference network $A|_{V'} = (V', \Pi|_{V'})$ where $\Pi|_{V'} = \{\pi'_s\}_{s \in V'}$ is defined by setting π'_s to be the linear order on $L(V')$ which keeps the relative ordering of all members of V' , i.e., for all $s, u, v \in V'$, $u \succ_{\pi'_s} v \iff u \succ_{\pi_s} v$. We say that A' is EMBEDDED into A if $A' = A|_{V'}$ for some $V' \subset V$.

Axiom 8 (Embedding (Emb)). *If $A' = (V', \Pi')$ is embedded into $A = (V, \Pi)$ and $\pi_i(j) = \pi'_i(j)$ for all $i, j \in V'$ then*

$$\mathcal{C}(A') = \mathcal{C}(A) \cap 2^{V'}.$$

In other words, if a network (V', Π') is embedded into a larger network (V, Π) in such a way that, with respect to the preferences in the larger network, the members of the smaller network prefer each other over everyone else, then the set of communities in the larger network which are subsets of V' is identical to the set of communities in the smaller network.

Note that, in contrast to the first seven axioms, which refer to a fixed finite ground set V , the last axiom links different ground sets to each other. Strictly speaking, a community rule \mathcal{C} is therefore not just one function $\mathcal{C} : (V, \Pi) \mapsto 2^{2^V - \{\emptyset\}}$, but a collection of such functions, one for each finite set V contained in some countable reference set, say the natural numbers⁴ \mathbb{N} . In a similar way, preference aggregation is not defined by a single function $F : L^*(V) \rightarrow \overline{L(V)}$ but by a set of such functions, one for each finite V contained in the reference set. However, when we define preference aggregation, we usually define it for a fixed V , leaving the dependence on V implicit.

Note also that together, Axioms Anonymity and Embedding imply the isomorphism invariance discussed in the introduction.

2.3 Properties of Social Choice Axioms

Before we begin to study the properties induced by social choice axioms, we look at the properties that fixed point rules have without any further assumptions. To this end, we will define two properties of a community rule \mathcal{C} .

Property 1 (Independence of Outside Opinions (IOO)). *A community function \mathcal{C} satisfies Independence of Outside Opinions if, for all subsets $S \subseteq V$ and all pairs of preference profiles Π, Π' on V such that $\pi'_s = \pi_s$ for all $s \in S$, we have that*

$$S \in \mathcal{C}(V, \Pi') \iff S \in \mathcal{C}(V, \Pi).$$

Property IOO simply states that the preferences of outsiders cannot influence whether or not a subset is a community. It turns out that this property (and one of our Axioms) is always satisfied by any fixed-point community rule.

Proposition 2. *All fixed-point rules satisfy Independence of Outside Opinions and World Community.*

⁴While we use the embedding axiom to make statements about subsets of a given ground set V , see, e.g., Propositions 5 and 7 below, we never use that we can embed a given preference network into an even larger one. Therefore, all results of this paper, except for those involving complexity statements, hold if one restricts oneself to a finite set V_0 , and only considers preference networks defined on subsets $V \subset V_0$.

Proof. Clearly, any fixed point rule satisfies IOO since the preferences of outsiders are entirely ignored when deciding if a subset constitutes a fixed point. The axiom WC is satisfied vacuously, because it involves looking at all $v \in V - V$. \square

Turning now to social choice axioms, we must first formally define the axioms informally described in Section 1.2. To this end, we need the notion of an election, which will be defined as a triple (V, F, S) where V and S are finite sets (called the set of candidates and voters, respectively), and $F : L(V)^* \rightarrow \overline{L(V)}$ is a preference aggregation function.

Social Choice Axiom 1 (Unanimity (U)). *An election (V, F, S) satisfies Unanimity if, for all preference profiles, $\Pi_S = \{\pi_s : s \in S\} \in L(V)^S$ and all pairs of candidates, $\{i, j\} \subseteq V$,*

$$\pi_s(i) > \pi_s(j), \forall s \in S \implies F(\Pi_S)(i) > F(\Pi_S)(j).$$

The question then is: what properties capture the intuition behind Unanimity and how do they relate to this social choice axiom? To answer this, we define the following two properties of a community function \mathcal{C} .

Property 2 (Pareto Efficiency (PE)). *A community function, \mathcal{C} , is Pareto Efficient if, for a given preference network A and a given community $S \in \mathcal{C}(A)$, it is the case that for all $u \in S$, $v \notin S$, there is a $s \in S$ such that $u \succ_{\pi_s} v$.*

Property 3 (Clique (Cq)). *A community function \mathcal{C} satisfies the Clique Property if for all $A = (V, \Pi)$,*

$$u \succ_{\pi_s} v, \forall u, s \in S, \forall v \notin S \implies S \in \mathcal{C}(A).$$

Property Pareto Efficiency is a negative property that states that subsets in which a non-member is preferred to a member by everyone inside the subset, should not be a community. In contrast, Clique is a positive Property, in that it states that a completely self-loving group (i.e., a clique) must be a community.

It turns out that both of these properties are implied by Unanimity.

Proposition 3. *Fix V and a preference aggregation function F , and let \mathcal{C}_F be the fixed point rule with respect to F . If all elections (V, F, S) with $S \subsetneq V$ satisfies Unanimity, then \mathcal{C}_F satisfies the properties Pareto Efficiency and Clique.*

Proof. Fix a preference network $A = (V, \Pi)$.

First, let us show that \mathcal{C}_F satisfies Pareto Efficiency. Assume otherwise. In this case there must be a community $S \subsetneq V$ such that for some $s \in S$ and $j \notin S$, everyone in S prefers j to s . However, this implies that j must be ranked higher than s in $F(\Pi_S)$ by Unanimity. By the pigeon hole principle this implies that the elements of S cannot occupy the first $|S|$ positions of this preference aggregation, and therefore S is not a community.

Now to show that \mathcal{C}_F satisfies the Clique Property, assume $S \subsetneq V$ is a clique ($\forall i, j \in S$ and $k \notin S$, $j \succ_{\pi_i} k$). Then all elements of S are preferred by all members of S to all members of $V - S$ and therefore must appear in the first $|S|$ slots of $F(\Pi_S)$ by Unanimity. This then implies that S is a community as required. \square

Social Choice Axiom 2 (Non-Dictatorship (ND)). *An election (V, S, F) is Non-Dictatorial if there exists no dictator, i.e., no voter $i \in S$ such that $F(\Pi_S) = \pi_i$ for all preference profiles $\Pi_S \in L(V)^S$.*

Instead of showing properties implied by ND as we did with **Unanimity**, we do the inverse, and show that a dictatorship violates some of our axioms.

Proposition 4. *Fix V and a preference aggregation function F . If \mathcal{C}_F , the fixed point rule with respect to F , satisfies **Group Stability** or **Anonymity**, then all elections (V, F, S) with $S \subset V$ and $1 < |S| < |V|$ satisfy **Non-Dictatorship**.*

Proof. Assume (V, F, S) is dictatorial, with dictator $s \in S$. Let π_s be such that all members of S are ranked above those outside of S . Because s is a dictator, we have that S is a community ($S \in \mathcal{C}_F$). Additionally let every other member of S rank some non-member $v \notin S$ above s .

However, if \mathcal{C}_F satisfies **Group Stability**, S cannot be a community. Furthermore, if \mathcal{C}_F satisfies **Anonymity**, if the preferences of any two members of S are swapped, S should remain a community. However, if s swaps with any other member of S , v will be ranked above s in the aggregate preference and thus S cannot be a community. \square

The last of the three social choice axioms, **Independence of Irrelevant Alternatives**, simply states that the aggregate relation between any two pairs of candidates should not depend on the preferences for any other candidate.

Social Choice Axiom 3. (Independence of Irrelevant Alternatives) *An election (V, F, S) satisfies Independence of Irrelevant Alternatives (IIA) if for all preference profiles, $\Pi_S, \Pi'_S \in L(V)^S$ and all candidates $a, b \in V$ we have that*

$$(\forall s \in S, a \succ_{\pi_s} b \Leftrightarrow a \succ_{\pi'_s} b) \implies (a \succ_{F(\Pi_S)} b \Leftrightarrow a \succ_{F(\Pi'_S)} b).$$

This axiom can reasonably be considered the strongest of the three, in that it says that the aggregate preference between two candidates does not even depend on the preferences voters have between either of the two and some other candidate. We will demonstrate this strength by proving an impossibility result involving modest assumptions about the fixed point rule of an aggregation function that satisfies IIA.

Theorem 1. *Let F be an aggregation function such that the fixed point rule with respect to F satisfies the **Clique Property** and the **Group Stability Axiom**. Then no election (V, F, S) with $S \subseteq V$ and $1 < |S| < |V|$ satisfies IIA.*

Proof. Let $S \subseteq V$ such that $1 < |S| < |V|$. Assume that the election (V, F, S) satisfies IIA, and the resulting fixed point rule \mathcal{C}_F satisfies **Cq** and **GS**. We will first show that the election (V, F, S) must satisfy **Unanimity**.

In the following preference profiles, $\Pi, \Pi', \Pi'' \in L(V)^S$, we assume that every member of S has the same preference, π, π' , and π'' respectively. First, let π rank all members of S above non-members. By the **Clique Property**, $S \in \mathcal{C}_F(A)$ and thus

$$\forall s \in S, v \notin S, s \succ_{F(\Pi)} v. \quad (1)$$

Thus, by IIA, if $s \in S$ is unanimously preferred to $v \notin S$, s must be strictly preferred to v in the aggregate preference.

Now let π' be the same as π only with the least preferred member of S , s' , and the most preferred non-member, v' , switched in rank. By the partial **Unanimity** property (1), in the

aggregate $F(\Pi')$, all members of $S - \{s'\}$ are preferred to all $v \notin S$, and all members of S are preferred to all $v \in V - S - \{s'\}$. On the other hand, by GS, $S \notin \mathcal{C}_F(\Pi')$, which is only possible is if $v' \succeq_{F(\Pi')} s'$. Applying the partial Unanimity property once more yields the following two statements:

$$\forall s \in S - \{s'\}, s \succ_{F(\Pi')} s' \quad \text{and} \quad \forall v \notin S \cup \{v'\}, v' \succ_{F(\Pi')} v,$$

and by IIA, this in turn implies

$$\forall s \in S - \{s'\}, s \succ_{F(\Pi)} s' \quad \text{and} \quad \forall v \notin S \cup \{v'\}, v' \succ_{F(\Pi)} v. \quad (2)$$

By IIA, this means that for any two members or two non-members if one is unanimously preferred to the other, then it must be strictly preferred in aggregate preference. Indeed, consider, e.g., $s, s' \in S$ and a profile $\tilde{\Pi}_S$ such that $s \succ_{\tilde{\Pi}_i} s'$ for all $i \in S$. Choose Π in such a way that every member has the same profile, s' has rank $|S|$ and $s \succ_{\pi_i} s'$ for all $i \in S$. By IIA, $s \succ_{F(\tilde{\Pi})} s' \iff s \succ_{F(\Pi)} s'$, so by (2), s is preferred to s' in aggregate.

Finally, consider π'' where v' is switched with the second lowest ranked member, s'' . By the above additional partial Unanimity property, s' must be strictly preferred to s'' in the aggregate preference $F(\Pi'')$, and therefore v' must be strictly rather than weakly preferred to s'' in the aggregate preference. Thus, again by IIA, if a non-member, $v \notin S$, is unanimously preferred to a member $s \in S$, v must be strictly preferred to s in the aggregate preference. Taken together, these three partial Unanimity properties, constitute Unanimity.

Since the election (V, F, S) satisfies both IIA and Unanimity, by Arrow's Impossibility Theorem [2] it must be a dictatorship, contradicting Proposition 4. \square

2.4 Additional Properties of Axioms

Here we state some additional properties of interest that community rules (not necessarily fixed point rules) have when they satisfy one or more of our main axioms.

Proposition 5. *Let \mathcal{C} be a community rule that satisfies the World Community and Embedding Axioms. Then \mathcal{C} must also satisfy the Cliques Property.*

Proof. Let $A = (V, \Pi)$ be a preference network and S be a clique (every member of S prefers S to $V - S$). By World Community, we have that $S \in \mathcal{C}((S, \Pi|_S))$ and by Embedding we have $\mathcal{C}(A) \cap 2^S = \mathcal{C}((S, \Pi|_S))$. Therefore S is a community. \square

Proposition 6. *Any community rule \mathcal{C} that satisfies Monotonicity must satisfy Independence of Outside Opinions.*

Proof. Let $A = (V, \Pi)$ be a preference network. Axiom Mon features an alternative preference profile Π' stating that if Π' satisfies certain properties and S is a community for (V, Π') , then S must be a community (V, Π) . Because the axiom places no restrictions on the preferences of voters from $V - S$, the rule \mathcal{C} must satisfy IOO. \square

Property 4 (Outsider Departure (OD)). *A community rule \mathcal{C} satisfies the Outsider Departure Property if for a given preference network $A = (V, \Pi)$, community $S \in \mathcal{C}(A)$, and outsider $v \notin S$, we have that $S \in \mathcal{C}(V - \{v\}, \Pi|_{V - \{v\}})$.*

Proposition 7. *A community rule, \mathcal{C} , that satisfies the Monotonicity and Embedding Axioms must also satisfy the Outsider Departure Property.*

Proof. Let $A = (V, \Pi)$ be a preference network, $S \in \mathcal{C}(A)$ a community, and $v \notin S$ an outsider. Consider the preference profile Π' that ranks v at the end of everyones preference. By Mon, $S \in \mathcal{C}(V, \Pi')$. Furthermore, since Π' satisfies the setup for Embedding, we also have $S \in \mathcal{C}(V - \{v\}, \Pi'|_{V - \{v\}})$. However, $\Pi'|_{V - \{v\}} = \Pi|_{V - \{v\}}$ since Π and Π' only differ in the placement of v . Therefore we have $S \in \mathcal{C}(V - \{v\}, \Pi|_{V - \{v\}})$. \square

Proposition 8. *If a community rule satisfies the Group Stability and Self-Approval Axioms it must satisfy the Pareto Efficiency Property.*

Proof. Let S be a community.

Case 1: $|S| = 1$. By Self-Approval, the one member s must rank herself above all outsiders and therefore satisfies PE.

Case 2: $|S| > 1$. Choose $G \subset S$ such that G is a singleton $\{s'\}$. By Group Stability, for all outsider singletons $\{g'\} \subseteq V - S$ and bijections $(f_i : \{s'\} \rightarrow \{g'\}, i \in S - G)$ there exists an $s \in S - G$ such that $s' \succ_{\pi_s} f_s(s')$. Since it is clear that $f_s(s') = g'$, s provides the necessary witness for s' and S satisfies PE. \square

3 Aggregation Based Communities Rules

We now examine several examples of aggregation based community rules through the lens of our axiomatic framework. In Section 3.1, we focus on a what we call weighted fixed-point rules, starting with the B³CT community function from [3]. We show that it violates both Axioms Monotonicity and Group Stability. The violation of the monotonicity axiom was initially somewhat of a surprise and rather counterintuitive to us. This violation is illustrative of the subtlety of community rules; indeed, it helped us to identify a weaker monotonicity property that the B³CT function satisfies. We then show that the fixed-point community rule based on any Borda-count-like voting function is inconsistent with either the Group Stability axiom or the Clique property. This impossibility result and Theorem 1 illustrate some basic limitations of fixed-point community rules. Next, we study the properties of the harmonious community function in Section 3.2. We will show that it can be obtained by preference aggregation, and that it obeys all of our axioms except for Axiom GS. It does, however, satisfy a weaker version of this axiom, see Theorem 4. In our final subsection, Section 3.3, we compare the three rules Borda voting, B³CT voting, and the harmonious rule.

3.1 Weighted Fixed Point Rules

This section focuses on a class of community rules that lie in between general fixed point rules and the B³CT community rule, which we call *weighted fixed point rules*. First, we will look at some of the properties of the B³CT rule as a particular case of a weighted fixed point rule.

Theorem 2. *The B³CT community rule, \mathcal{C}_{B^3CT} , does not satisfy Monotonicity or Group Stability. It satisfies all other axioms, as well as Properties Pareto Efficiency and Clique.*

Proof. Directly from the definition of the B³CT voting function ϕ_S^Π , \mathcal{C}_{B^3CT} satisfies Axioms A, WC, Emb, and Properties PE and Cq. Suppose \mathcal{C}_{B^3CT} does not satisfy SA. Then, there exists a preference network $A = (V, \Pi)$, $S \in \mathcal{C}_{B^3CT}(A)$, $T \subseteq V - S$, and a tuple of bijections $(f_s : S \rightarrow T)$ such that for all $s, u \in S$, $u \prec_{\pi_s} f_s(u)$. It follows that $\forall s \in S$, the numbers of votes cast by s for S according to ϕ_S^Π is less than the numbers of votes that s casts for T . Summing up the votes from S , the average votes that members of T receive is larger than the average votes that members of S receive, contradicting the assumption that everyone in S receives more votes than everyone in T . Thus, \mathcal{C}_{B^3CT} satisfies SA.

To show \mathcal{C}_{B^3CT} satisfies Axiom CRM, consider S , Π and Π' as in Axiom CRM. By the coherence assumption for members, there exists $\sigma \in L(S)$ such that for $s_1, s_2 \in S$, for all $s \in S$, $s_1 \succ_{\pi'_s} s_2$ if and only if $s_1 \succ_\sigma s_2$.

Let s^* denote the least preferred elements of S according to σ . By the assumption that $\pi_s(v) = \pi'_s(v)$ for all $s \in S, v \in V - S$, we have that $\pi_s(V - S) = \pi'_s(V - S)$, and hence also that $\pi_s(S) = \pi'_s(S)$. But this implies that for all $u \in S$

$$\phi_S^\Pi(u) = \sum_{s \in S} 1_{\pi_s(u) \leq |S|} \geq \sum_{s \in S} 1_{\pi_s(S) \subseteq [1:S]} = \sum_{s \in S} 1_{\pi'_s(S) \subseteq [1:S]} = \sum_{s \in S} 1_{\pi'_s(s^*) \leq |S|} = \phi_S^{\Pi'}(s^*).$$

If $S \in \mathcal{C}_{B^3CT}(V, \Pi')$, then s^* receives more votes from Π'_S than every $v \in V - S$, and the number of votes v receives from Π_S is the same as the number of votes it receives from Π'_S . On the other hand, for all $u \in S$, the number of votes u receives from Π_S is at least the number of votes s^* receives from Π'_S , implying that $S \in \mathcal{C}_{B^3CT}(V, \Pi)$. We can similarly show that \mathcal{C}_{B^3CT} satisfies Axiom CRNM.

Let $V = [1 : 6]$, $S = [1 : 3]$, let $\Pi = (\pi_1, \dots, \pi_6)$ be the preference profile

$$\begin{aligned} \pi_1 &= [142356], & \pi_2 &= [253416], & \pi_3 &= [631425] \\ \pi_4 &= [456123], & \pi_5 &= [156423], & \pi_6 &= [165423] \end{aligned}$$

and let Π' be the preference profile

$$\begin{aligned} \pi'_1 &= [142356], & \pi'_2 &= [234516], & \pi'_3 &= [314625] \\ \pi'_4 &= \pi_4, & \pi'_5 &= \pi_5, & \pi'_6 &= \pi_6. \end{aligned}$$

Then $S = [1 : 3] \in \mathcal{C}_{B^3CT}(V, \Pi)$, as each members of S receives two votes while everyone in $[4 : 6]$ receives only one vote. However, in violation of Axiom Mon, S is no longer a B³CT community w.r.t Π' , since 4 now receives three votes, one more than 1, 2 and 3.

Note also $T = (1, 5, 6) \in \mathcal{C}_{B^3CT}(V, \Pi)$. Let $G = \{5, 6\} \subset T$ and $G' = (2, 4) \subset V - T$. As member 1 prefers 2 to 5 and 4 to 6, T does not satisfy Group Stability. \square

Note that the same analysis shows that \mathcal{C}_{B^3CT} does not satisfy the Outsider Departure Property. In the example above, if member 5 leaves the system, then member 4 will receive 2 votes from $S = \{1, 2, 3\}$, and hence S is no longer a \mathcal{C}_{B^3CT} -community.

Even though \mathcal{C}_{B^3CT} does not satisfy Mon, it does enjoy the following monotonicity property.

Property 5 (Outsider Respecting Monotonicity). *If S is a community of a preference network $A = (V, \Pi)$, then S remains a community of (V, Π') for any Π' such that (1) $u \succ_{\pi_s} t \Rightarrow u \succ_{\pi'_s} t$, $\forall u, s \in S, t \in V$, and (2) $v \succ_{\pi_s} v' \Rightarrow v \succ_{\pi'_s} v'$, $\forall v, v' \in V - S, s \in S$.*

We now analyze the fixed point rule defined by the family of aggregation functions, such as Borda count and B³CT voting, that derive a cardinal social preference from ordinal individual preferences.

Let W be a sequence of weight vectors $w^i \in \mathbb{R}^n$, $W = (w^1, w^2, \dots)$, where n is the number of elements in V . For a non-empty finite $S \subset \mathbb{N}$ and $\Pi_S \in L(V)^S$ define the aggregate preference $F_W(\Pi_S)$ on V by

$$i \succ_{F(\Pi_S)} j \iff \sum_{s \in S} w_{\pi_s(i)}^{|S|} > \sum_{s \in S} w_{\pi_s(j)}^{|S|}.$$

In other words, $i \succ j$ in the aggregate iff the total weight of the votes i receives from S is larger than the total weight of the votes j receives from S , where a vote in position p gets weight $w_p^{|S|}$.

In B³CT, w^k is the vector of k ones followed by $(n - k)$ zeros⁵, while Borda count uses $w^k = (n, n - 1, \dots, 1)$ for all k .

Definition 8 (Weighted Fixed Point Rule). *For a sequence of vectors $W = (w^1, w^2, \dots)$ in \mathbb{R}^n , \mathcal{C}_W is the fixed point rule with respect to F_W .*

Proposition 9. *Weighted fixed-point rules satisfy Axiom Anonymity. They satisfy Outsider Respecting Monotonicity if $w_i^k \geq w_j^k$ for all $k \in [1 : n - 1]$ and $i \leq j$, and they satisfy the Clique Property if and only if for all $k \in [1 : n - 1]$ the weight vector w^k is such that $w_i^k > w_j^k$ for $i \leq k$ and $j > k$.*

Proof. The proof of the first two statements and the “if” part of the third follow directly from the definitions. To see the “only if” part of the third statement, consider k, i, j such that $w_i^k \leq w_j^k$, and let $S, \Pi \in L(V)$ be such that $|S| = k$, $\pi_s \leq k$ for all $s, u \in S$, and $\pi_s(v) = \pi_t(v)$ for all $s, t \in S, v \in V$. Then S satisfies the condition of the Property Cq, but it is not a community. To see this, choose $v \in S$ and $v' \notin S$ such that $\pi_s(v) = i$ and $\pi_s(v') = j$. Then $\sum_{s \in S} w_{\pi_s(v)}^k \leq \sum_{s \in S} w_{\pi_s(v')}^k$, showing that S is not a community. \square

Together with Proposition 5, the next theorem implies that there is no weighted fixed point rule that satisfies the Group Stability, World Community and Embedding Axioms.

Theorem 3. (IMPOSSIBILITY OF WEIGHTED AGGREGATION SCHEMA) *Weighted Fixed Point Rules are inconsistent with either the Group Stability Axiom or the Clique Property.*

Proof. Let $A = (V, \Pi)$ be a preference network, $S \subset V$, and \mathcal{C}_W a weighted fixed point rule satisfying the the Clique Property. Throughout the the proof, we will take

$$V = \{a, b, c, d, e\} \quad \text{and} \quad S = \{a, b, c\},$$

and consider preference profiles such that S violates Group Stability. In order for \mathcal{C}_W to obey the Axiom GS, we would need the weight vector $w^3 \in \mathbb{R}^5$ to be such that $S \notin \mathcal{C}(V, \Pi)$ for all Π considered in this proof. Our goal is to show that this will lead to a contradiction. We start under the assumption that the weights are decreasing, i.e., in addition to the already established

⁵The rule $\mathcal{C}_{\text{B}^3\text{CT}}$ does not specify what the weight w^k should be for $k > n$ since preferences with more voters than alternatives do not occur when determining communities – so we are free to define it arbitrarily, say $w_i^k = 1$ for all i if $k > n$.

fact that $w_i^3 > w_j^3$ when $i = 1, 2, 3$ and $j = 4, 5$ (since \mathcal{C}_W satisfies the the Clique Property), we will first assume that $w_1^3 \geq w_2^3 \geq w_3^3$ and $w_4^3 \geq w_5^3$.

Consider the following scenario:

$$\pi_a = [adebc], \pi_b = \pi_c = [abcde].$$

Since a prefers d and e over b and c , S is not group stable and hence cannot be a community. By our assumption that $w_1^3 \geq w_2^3 \geq w_3^3 > w_4^3 \geq w_5^3$, we have that $a \succ_{F_W(\Pi_S)} b \succeq_{F_W(\Pi_S)} c \succ_{F_W(\Pi_S)} e$ and $b \succ_{F_W(\Pi_S)} d$. Therefore the only way S cannot be a community is that $d \succeq_{F_W(\Pi_S)} c$, i.e.,

$$w_2^3 + 2w_4^3 \geq 2w_3^3 + w_5^3.$$

Notice that this implies that we cannot have both $w_2^3 = w_3^3$ and $w_4^3 = w_5^3$.

Now consider a modified preference profile:

$$\pi'_a = \pi'_b = [abdce], \pi'_c = [caebd].$$

In this profile a and b prefer d over c , so again S violates GS and hence cannot be a community. On the other hand, we now have $a \succ_{F_W(\Pi'_S)} b$, $b \succeq_{F_W(\Pi'_S)} d \succ_{F_W(\Pi'_S)} e$. Thus we must have either $b \sim_{F_W(\Pi')} d$ or $d \succeq_{F_W(\Pi'_S)} c$. The former, however, implies $w_2^3 = w_3^3$ and $w_4^3 = w_5^3$ and is hence a contradiction. Therefore the latter must be true which implies

$$2w_3^3 + w_5^3 \geq w_1^3 + 2w_4^3.$$

This brings us to the final preference profile:

$$\pi''_a = [abdce], \pi''_b = [dcabe], \pi''_c = [cbaed].$$

Again a and b prefer d to c , so the profile violates GS, and hence again can't be a community. Now $a \succ_{F_W(\Pi'')} c \succeq_{F_W(\Pi'')} b$ and $d \succ_{F_W(\Pi'')} e$, showing that for S not to be a community, we must have $d \succeq_{F_W(\Pi'')} b$, which gives

$$w_1^3 + w_3^3 + w_5^3 \geq 2w_2^3 + w_4^3.$$

Defining $d_i = w_i^3 - w_{i-1}^3$, we can write the bounds obtained so far as

$$d_4 \leq d_3 + d_5$$

$$d_2 + d_3 + d_5 \leq d_4$$

$$d_3 + d_4 + d_5 \leq d_2.$$

Chaining up these three bounds, we get

$$d_3 + d_5 \geq d_4 \geq d_2 + d_3 + d_5 \geq d_3 + d_4 + d_5 + d_3 + d_5 = 2(d_3 + d_5) + d_4,$$

contradicting our assumption $d_i \geq 0$ and the fact that \mathbf{Cq} implies $d_4 > 0$.

To relax the constraint that the weights are ordered, we observe that all three profiles considered in the proof are such that, under arbitrary permutations of the first three and the last two positions, S still violates GS. In other words, for any permutation σ of $[1 : 5]$ that leaves $[1 : 3]$ and $[4 : 5]$ invariant, S violates GS under the profiles $\{\sigma \circ \pi_s\}_{s \in S}$, $\{\sigma' \circ \pi_s\}_{s \in S}$, and $\{\sigma'' \circ \pi_s\}_{s \in S}$. Choosing the permutation in such a way that the weights $\tilde{w}_i^3 = w_{\sigma(i)}^3$ are ordered, we obtain the above three inequalities for the weights \tilde{w}_i^3 , leading again to a contradiction. \square

3.2 Properties of Harmonious Communities

In this subsection, we analyze the harmonious community function given by Definition 5. We first prove that it can be expressed in terms of a suitable preference aggregation function.

Proposition 10. *There exists a preference aggregation function $F_{\mathcal{H}} : L(V)^* \rightarrow \overline{L(V)}$ such that the harmonious community function \mathcal{H} is defined by a $F_{\mathcal{H}}$.*

Proof. Given V , a finite set S , and a preference profile $\Pi_S \in L(V)^S$, we consider the following directed graph $G_{\Pi_S} = (V, E_{\Pi_S})$ where $(i, j) \in E_{\Pi_S}$ if at least half of S prefers i to j . Note that if $|S|$ is an odd number, then G_{Π_S} is a *tournament graph*. If $|S|$ is an even number, then E_{Π_S} contains both (i, j) and (j, i) if exactly half of Π_S prefer i to j . G_{Π_S} is *total* since for all $i, j \in V$, either $(i, j) \in E_{\Pi_S}$ or $(j, i) \in E_{\Pi_S}$. Because G_{Π_S} is total, the graph \hat{G}_{Π_S} obtained from G_{Π_S} by contracting each strongly connected component into a single vertex is an *acyclic*, tournament graph. As a consequence, the graph \hat{G}_{Π_S} has exactly one Hamiltonian path that totally orders its vertices. Let (V_1, \dots, V_t) be the strongly connected components of G_{Π_S} , sorted by the order determined by the Hamiltonian path. The partition (V_1, \dots, V_t) of V then defines an ordered partition $F_{\mathcal{H}}(\Pi_S)$, with $V_i \succ_{F_{\mathcal{H}}(\Pi_S)} V_j$ iff $i \leq j$.

Next, we consider a subset $T \subset V$. It is then easy to check that if T is of the form $T = \cup_{j \leq i} V_j$ for some $i \in [1 : t]$, then for all $u \in T, v \in V - T$, a majority of S prefers u to v , and vice versa. Specializing to $S = T$, we see that \mathcal{H} is defined by the preference aggregation function $F_{\mathcal{H}}$. \square

Next we show that \mathcal{H} satisfies all axioms except for Group Stability.

Theorem 4. *The harmonious community function satisfies Axioms A, SA, Mon, Emb, WC, CRM, and CRNM, but it does not satisfy GS.*

Proof. Directly from the definitions, one easily checks that \mathcal{H} satisfies Axioms A, Mon, Emb and WC.

By a similar argument to the proof of Theorem 2, we can prove that \mathcal{H} satisfies SA: if $S \in \mathcal{H}(A)$ does not satisfy SA, then there exists a $T \subset V - S$ of the same size as S such that each $s \in S$ lexicographically prefers T over S . With the help of Proposition 1, this implies that, for each $s \in S$, there are at least $(1 + 2 + \dots + |S|)$ pairs $(u, v) \in S \times T$ such that s prefers v over u . Thus the number of triples (s, u, v) such that $s \in S$ prefers $v \in T$ over $u \in S$ is at least $|S|^2(|S| + 1)/2$. However, $S \in \mathcal{H}(A)$ implies that this number has to be strictly smaller than $|S|^3/2$.

To see that \mathcal{H} is consistent with Axiom CRNM, consider a preference profile Π, Π' as specified in Axiom CRNM. By the coherence assumption on non-members, there exists a linear order σ on $V - S$, such that $\forall i, j \in V - S$ and $\forall s \in S$, $i \succ_{\pi'_s} j \Leftrightarrow i \succ_{\sigma} j$. Let v^* be the most preferred element of σ . By the assumption that $\pi_s(u) = \pi'_s(u)$ for all $s, u \in S$, we have $\pi_s(S) = \pi'_s(S)$ and hence also $\pi_s(V - S) = \pi'_s(V - S)$. But this implies that for all $v \in V - S$,

$$\pi_s(v) \geq \min\{i \in \pi_s(V - S)\} = \min\{i \in \pi'_s(V - S)\} = \pi'_s(v^*).$$

We therefore have shown that for all $s, u \in S$ such that $u \succ_{\pi'_s} v^*$, we have that $u \succ_{\pi_s} v$ for all $v \in V - S$. Assume now that $S \in \mathcal{H}((V, \Pi'))$. Then for all $u \in S$, the majority of (Π', S) prefer u to v^* , which, as we just have shown, implies that for all $v \in V - S$, the majority of (Π, S)

prefer u to v , which in turn implies that $S \in \mathcal{H}((V, \Pi))$. We can similarly show that \mathcal{H} satisfies Axiom CRM.

The set T in the proof of Theorem 2 is also an example that \mathcal{H} violates Axiom GS. \square

While \mathcal{H} does not satisfy the GS Axiom, it satisfies the following weaker property.

Property 6. Weak Group Stability *For all preference profiles Π on V and all $S \in \mathcal{C}(V, \Pi)$, S is weakly group stable. Here a set $S \subset V$ is called weakly group stable if for all $G \subset S$, $G' \subset V - S$ s.t. $0 < |G| = |G'| \leq |S|/2$, and all bijections $(f : G \rightarrow G', i \in S - G)$ there exists $s \in S - G$, $u \in G$ such that $u \succ_{\pi_s} f(u)$.*

Note that the property is weaker than the GS Axiom in two ways: we restrict ourselves to groups G of size at most $|S|/2$, and we only allow for a global bijection f , rather than individual bijections f_s .

Proposition 11. *\mathcal{H} is weakly group stable, while the Borda count and the B^3CT rule are not.*

Proof. Consider a set $S \in \mathcal{H}(V, \Pi)$, subsets $G \subset S$ and $G' \subset V - S$ such that $0 < |G| = |G'| \leq |S|/2$, and a bijection $f : G \rightarrow G'$. For each $u \in G$ the majority of S prefer u to $f(u)$ (who is not a member of S), and since $|G| \leq |S|/2$, this implies that there must be at least one $s \in S - G$ such that s prefers u to $f(u)$, as required.

To give a counterexample for both Borda counting and the B^3CT rule, consider $V = [1 : 6]$, $G = [3 : 4]$ and $G' = [5 : 6]$, with preference profiles

$$\pi_1 = [125463], \pi_2 = [126354], \pi_3 = [341256], \pi_4 = [341256].$$

Then 1 and 2 prefer 5 over 4, and 6 over 3, but S is a community both with respect to B^3CT (where 1 and 2 get four votes, 3 and 4 get three votes, and 5 and 6 get only one vote), and with respect to Borda count (with counts 20, 16, 18, 16, 10, 8 for 1, \dots , 6, respectively). \square

Proposition 12. *\mathcal{H} satisfies IOO as well as Cq and the PE, but $F_{\mathcal{H}}$ does not satisfy U.*

Proof. By Proposition 2, \mathcal{H} satisfies IOO. To see that it does not satisfy U, let $V = \{a, b, c\}$, let $S = \{a, b\}$ and $\pi_a = (acb)$, $\pi_b = (bac)$. Then $a \succ_{\pi_s} c$ for all $s \in S$, and both $a \succ_{\pi_s} b$ and $b \succ_{\pi_s} c$ in half of S . Therefore $(ac), (cb), (bc), (ab), (ba) \in E_{\Pi_S}$. Thus, a, b, c belongs to the same connected component in G_{Π_S} , showing that S is not a harmonious community. To see that \mathcal{H} satisfies both Cq and PE in spite of the fact that it does not satisfy the assumptions of Proposition 3, we use Proposition 5 to infer Cq, and the observation that $S \in \mathcal{H}(A)$ implies that for any a pair of elements $(u \in S, v \notin S)$, the majority of S prefer u over v , proving PE. \square

3.3 Comparison of Borda voting, B^3CT voting, and the harmonious rule

In this subsection, we compare the fixed-point community rules that we have discussed so far: Borda voting, B^3CT voting, and the harmonious rule. While all three have their own appealing simplicity and intuition and all satisfy Axioms A, SA, Emb, WC, CRM, and CRNM, there are significant differences with respect to Axioms Mon and GS, and the Outsider Departure property.

- **Outsider Departure:** A harmonious community S remains a harmonious community when any outsider $v \notin S$ leaves the system since the departure does not alter any pairwise preferences. However, for a B³CT community S , the departure of an outsider can increase the votes for other outsiders enough to destabilize the B³CT community. In a similar way, one can see that the Borda count rule is also unstable to departure of an outsider.
- **Monotonicity:** The harmonious rule satisfies Axiom **Mon**. The other two only satisfy the weaker **Outsider Respecting Monotonicity** property⁶.
- **Group Stability:** The subset T in the proof of Theorem 2 is a community according to all these three community rules. But T violates **GS** because 1 prefers outsiders over 5 and 6, even though 5 and 6 prefer 1 over everyone else: Element 1 is an “*arrogant*” member of its community. All aggregation functions satisfying **Unanimity** seem to be prone to existence of “arrogant” members. The harmonious rule satisfies the stability of majority subgroup under a global bijection f , although the stability of the minority subgroup (or the majority subgroup with individual bijections f_s) may not be guaranteed. The fixed-point rule of Borda count and B³CT voting essentially have no guarantee of group stability.
- **Small World:** In general, we say a community function \mathcal{C} satisfies the **Small World** property if

$$S \in \mathcal{C}((V, \Pi)) \Leftrightarrow \forall U \subseteq V - S, |U| < |S|, S \in \mathcal{C}(S \cup U, \Pi|_{S \cup U}).$$

This Helly-type property [5] localizes the identification of a community. Note that the **Small World** property includes some form of **Outsider Departure** together with the property that every community is “locally” verifiable. One can easily show that the fixed-point rules of the Borda count or B³CT voting do not have the **Small World** property, while the harmonious rule enjoys the following stronger variant of the small world property

$$S \in \mathcal{H}((V, \Pi)) \Leftrightarrow \forall v \in V - S, S \in \mathcal{H}(S \cup \{v\}, \Pi|_{S \cup \{v\}}),$$

and hence the property given in (3).

4 Taxonomy of Community Rules

In this section, we characterize the taxonomy of the axiom-conforming community rules.

First, in Section 4.1, we define two rules, the **Clique Rule** and the **Comprehensive Rule**, which satisfy all axioms, and which are most selective and most comprehensive, respectively, in the sense that any rule which satisfies all axioms leads to a set of communities which contains all communities defined by the **Clique Rule** and is contained in the **Comprehensive Rule** (the statement that this is the case, Theorem 5, will be our main theorem in this subsection).

In the next subsection, Section 4.2, we then expand on this “**Taxonomy Theorem**”, and show that under the following natural intersection and union of community rules, the family of all community rules that satisfies all eight axioms forms a bounded lattice. We will use the following two set-theoretic operators of community functions to define these lattice structures.

⁶Again, we can use the profiles from the proof of Theorem 2 to show that the Borda count rule does not satisfy **Mon**.

Definition 9 (Operations over Community Rules). *For two community functions \mathcal{C}_1 and \mathcal{C}_2 , we define the intersection and union, $\mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{C}_1 \cup \mathcal{C}_2$, as the community functions which, for all preference networks A , respectively satisfy*

$$\begin{aligned}(\mathcal{C}_1 \cap \mathcal{C}_2)(A) &:= \mathcal{C}_1(A) \cap \mathcal{C}_2(A) \\ (\mathcal{C}_1 \cup \mathcal{C}_2)(A) &:= \mathcal{C}_1(A) \cup \mathcal{C}_2(A).\end{aligned}$$

4.1 From the Most Selective to Most Comprehensive Rule

We start with perhaps the simplest rule for communities that satisfies the Clique Property.

Rule 1 (Clique Rule (\mathcal{C}_{clique})). *A non-empty subset $S \subseteq V$ is a community of $A = (V, \Pi)$, if and only if $\forall u, s \in S, v \notin S, u \succ_{\pi_s} v$. We use \mathcal{C}_{clique} to denote the community function defined by this rule.*

Proposition 13. *\mathcal{C}_{clique} satisfies all Axioms.*

Proof. The (easy) proof is left as an exercise for the reader. \square

However, the clique rule appears to be too restrictive, since it has the following structural feature, which essentially rules out any non-trivial overlap of communities, while “Real-world” communities typically have non-trivial overlaps among themselves.

Proposition 14. *For any preference network A , if $S_1, S_2 \in \mathcal{C}_{clique}(A)$, then either $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$, or $S_2 \subset S_1$.*

Proof. Assume otherwise. By assumption, we can choose an element $s \in S_1 \cap S_2$. Without loss of generality assume $|S_1| \leq |S_2|$. Again by assumption, there exists an element $s' \in S_1$ and $s' \notin S_2$. By the definition of the \mathcal{C}_{clique} s must have s' in its top $|S_1|$ choices. However, this means that s' is also in the top $|S_2|$ choices for s , which violates the fact that S_2 is in $\mathcal{C}_{clique}(A)$. \square

Next we address the question of whether there are rules consistent with all axioms that admit overlapping communities. To address this question, we consider rules defined by community axioms.

Rule 2 (Axiom Based Community Rules). *For $X \in \{\text{GS}, \text{SA}\}$ let \mathcal{C}_X be the community rule defined by $A = (V, \Pi) \mapsto \mathcal{C}_X(A)$, where $\mathcal{C}_X(A)$ is the set of non-empty subsets $S \subset V$ such that S obeys axiom X .*

For example, \mathcal{C}_{GS} denote the community rule that $S \in \mathcal{C}_{GS}(A)$ if and only S enjoys the Group Stability Axiom.

The first part of our Taxonomy Theorem is a direct consequence of the following basic lemma.

Lemma 1. (INTERSECTION LEMMA: GS AND SA) *For $X \in \{\text{A}, \text{Mon}, \text{CRM}, \text{CRNM}, \text{WC}, \text{Emb}\}$, if \mathcal{C} satisfies Axiom X , then $\tilde{\mathcal{C}} = \mathcal{C} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA}$ satisfies Axioms X , GA and SA.*

Proof. \mathcal{C}_{GS} and \mathcal{C}_{SA} are both consistent with A, WC, and Emb, thus if \mathcal{C} satisfies Axiom $X \in \{\text{A}, \text{WC}, \text{Emb}\}$, then $\tilde{\mathcal{C}}$ remains consistent with Axiom X .

To see $\tilde{\mathcal{C}}$ satisfies Axiom Mon if \mathcal{C} satisfies Mon, choose Π, Π' such that, for all $u, s \in S$ and $v \in V$, $u \succ_{\pi'_s} v \implies u \succ_{\pi_s} v$. We need to show that if $S \in \tilde{\mathcal{C}}((V, \Pi'))$ then $S \in \tilde{\mathcal{C}}((V, \Pi))$.

Suppose this is not the case, then either (1) $S \notin \mathcal{C}_{GS}((V, \Pi))$ or (2) $S \notin \mathcal{C}_{SA}((V, \Pi))$. In Case (1), there exists $G \subset S$, $G' \subset V - S$, $|G| = |G'|$, and bijections $(f_s : S \rightarrow G' | s \in S - G)$ such that $\forall s \in S - G, \forall u \in G, u \prec_{\pi_s} f_s(u)$. Then by the condition stated in **Mon**, we have $u \prec_{\pi'_s} f_s(u)$, which shows $S \notin \mathcal{C}_{GS}(A')$. In Case (2), there exists $G' \subset V - S$, bijections $(f_s : S \rightarrow G' | s \in S)$ such that $\forall s, u \in S, u \prec_{\pi_s} f_s(u)$. Then by the condition stated in **Mon**, we have $u \prec_{\pi'_s} f_s(u)$, which implies that $S \notin \mathcal{C}_{GS}(A')$.

Suppose \mathcal{C} satisfies Axiom **CRM**. Consider Π, Π' as specified in Axiom **CRM**. Given $s \in S$, the profiles π_s and π'_s are then assumed to be identical on $V - S$, implying in particular that $\pi_s(V - S) = \pi'_s(V - S)$, and hence also that $\pi_s(S) = \pi'_s(S)$. Furthermore, by the coherence assumption for members, there exist $\sigma \in L(S)$ such that $\forall u_1, u_2, s \in S, u_1 \succ_{\pi'_s} u_2$ iff $u_1 \succ_{\sigma} u_2$. We need to show that if $S \in \tilde{\mathcal{C}}((V, \Pi'))$ then $S \in \tilde{\mathcal{C}}(A)$. Suppose this is not the case, then either (1) $S \notin \mathcal{C}_{GS}(A)$ or (2) $S \notin \mathcal{C}_{SA}(A)$.

In Case (1), there exists $G \subset S$, $G' \subset V - S$, $|G| = |G'|$, a set of bijections $(f_s : G \rightarrow G' | s \in S - G)$, such that $\forall s \in S - G, u \in G, u \prec_{\pi_s} f_s(u)$. Let $T \subset S$ be the set of $|G|$ least preferred elements by σ . We now show that there exists bijections $(f'_s : T \rightarrow G' | s \in S)$ such that $\forall s \in S - T, u \in T, u \prec_{\pi'_s} f'_s(u)$, which would imply that $S \notin \mathcal{C}_{GS}((V, \Pi'))$.

Let us denote T by $T = \{t_1, \dots, t_{|T|}\}$ such that $t_i \prec_{\sigma} t_{i+1}$. Fix an $s \in S - T$, and let us denote G by $G = \{g_1, \dots, g_{|T|}\}$ such that $g_i \prec_{\pi_s} g_{i+1}$, and denote G' by $G' = \{g'_1, \dots, g'_{|T|}\}$ such that $g'_i \prec_{\pi_s} g'_{i+1}$. By Proposition 1, we then have that $g_i \prec_{\pi_s} g'_i$ for all $i = 1, \dots, |T|$. In other words, $\pi_s(g_i) > \pi_s(g'_i)$. We define f'_s by mapping t_i to g'_i . Note that the positions of the preferences rankings of S as a set are the same in π'_s and π_s . Because T is the set of $|G|$ least preferred elements of S , we have $\pi'_s(t_i) > \pi_s(g_i)$. Since $\pi'_s(g'_i) = \pi_s(g'_i)$ it then follows that $\pi'_s(t_i) > \pi_s(g_i) > \pi_s(g'_i) = \pi_s(g'_i)$. Thus, $t_i \prec_{\pi'_s} g'_i$, and consequently, $S \notin \mathcal{C}_{GS}((V, \Pi'))$. In Case (2), there exists $G' \subset V - S$ and a set of bijections $(f_s : S \rightarrow G' | s \in S)$, such that $\forall s, u \in S, u \prec_{\pi_s} f_s(u)$. By the similar argument as in Case (1) (by setting $T = S$), we can show that there exists bijections $(f'_s : S \rightarrow G' | s \in S)$ such that $\forall s \in S, u \in S, u \prec_{\pi'_s} f'_s(u)$, which implies that $S \notin \mathcal{C}_{GS}((V, \Pi'))$. Thus, $\tilde{\mathcal{C}}$ satisfies Axiom **CRM**.

We can similarly prove that $\bar{\mathcal{C}}$ satisfies **CRNM** if \mathcal{C} satisfies it.

Finally, by definition, $\mathcal{C} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA}$ satisfies **GS** and **SA**. □

Rule 3. (**COMPREHENSIVE COMMUNITY RULE**) *For a preference network $A = (V, \Pi)$, a non-empty $S \subseteq V$ is a community according to $\mathcal{C}_{comprehensive}$ if and only if S satisfies both Group Stability and Self-Approval axioms. In other words,*

$$\mathcal{C}_{comprehensive} := \mathcal{C}_{GS} \cap \mathcal{C}_{SA}.$$

We now prove that $\mathcal{C}_{comprehensive}$ is indeed the most comprehensive community rule that satisfies all Axioms.

Theorem 5 (Taxonomy: Lattice Top and Bottom). *$\mathcal{C}_{comprehensive}$ satisfies all Axioms. Moreover, for any community function \mathcal{C} that satisfies all Axioms, for every preference network $A = (V, \Pi)$*

$$\mathcal{C}_{clique}(A) \subseteq \mathcal{C}(A) \subseteq \mathcal{C}_{comprehensive}(A). \quad (3)$$

Proof. $\mathcal{C}_{all}(A) = 2^V - \{\emptyset\}$ satisfies Axioms **A**, **Mon**, **CRM**, **CRNM**, **WC** and **Emb**. Since $\mathcal{C}_{comprehensive} = \mathcal{C}_{all} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA}$, by the Intersection Lemma, $\mathcal{C}_{comprehensive}$ satisfies all Axioms.

On the other hand, by Proposition 5, any rule which satisfies WC and Emb, must satisfy the Cliques Property, so for any \mathcal{C} that satisfies all axioms, $\mathcal{C}_{clique}(A) \subseteq \mathcal{C}(A) \subseteq \mathcal{C}_{GS}(A) \cap \mathcal{C}_{SA}(A)$. Thus $\mathcal{C}_{clique}(A) \subseteq \mathcal{C}(A) \subseteq \mathcal{C}_{comprehensive}(A)$. \square

Theorem 5 shows that $\mathcal{C}_{comprehensive}$ and \mathcal{C}_{clique} are the most inclusive and the most selective function, respectively, that satisfies all axioms. While it is very easy to determine whether a subset in a preference network satisfies Property Clique, in Section 5 we demonstrate that $\mathcal{C}_{comprehensive}$ is highly “non-constructive” by showing that the decision problem for determining whether a subset in a preference network satisfies Axiom Self-Approval or Group Stability is coNP-complete.

4.2 The Lattice Structure of Community Rules

The Intersection Lemma provides us with a tool for exploring the taxonomy of community rules. In this subsection, we continue this exploration and make it more systematic using two lattice structures enjoyed by the community-rule taxonomy.

Theorem 6 (Taxonomy: Lattice Structures of Community Rules). *Let \mathcal{C} denote the family of all community rules that satisfies all eight axioms. Let \mathcal{C}_B be a superset of \mathcal{C} that denotes the family of all community rules that satisfies Axioms A, Mon, CRM, CRNM, WC, Emb.*

1. *The algebraic structure $\mathcal{T} = (\mathcal{C}, \cup, \cap, \mathcal{C}_{clique}, \mathcal{C}_{comprehensive})$ forms a bounded lattice with \mathcal{C}_{clique} as the lattice’s bottom and $\mathcal{C}_{comprehensive}$ as the lattice’s top.*
2. *The algebraic structure $\mathcal{T}_B = (\mathcal{C}_B, \cup, \cap, \mathcal{C}_{clique}, \mathcal{C}_{all})$ forms a bounded lattice with \mathcal{C}_{clique} as the lattice’s bottom and \mathcal{C}_{all} as the lattice’s top.*

Proof. First, by definition, the two operations \cap and \cup over the community functions are both commutative and associative. One can easily show that the two operations \cap and \cup satisfy the *absorption property*, that is, for any two $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{C}$

$$\begin{aligned}\mathcal{C}_1 \cup (\mathcal{C}_1 \cap \mathcal{C}_2) &= \mathcal{C}_1. \\ \mathcal{C}_1 \cap (\mathcal{C}_1 \cup \mathcal{C}_2) &= \mathcal{C}_1.\end{aligned}$$

For example, to see the first one, for any affinity network A , we have

$$(\mathcal{C}_1 \cup (\mathcal{C}_1 \cap \mathcal{C}_2))(A) = \mathcal{C}_1(A) \cup (\mathcal{C}_1 \cap \mathcal{C}_2)(A) = \mathcal{C}_1(A) \cup (\mathcal{C}_1(A) \cap \mathcal{C}_2(A)) = \mathcal{C}_1(A).$$

To complete the proof that \mathcal{T} and \mathcal{T}_B are lattices, we need to prove that \mathcal{T} and \mathcal{T}_B are closed under \cap and \cup . We organize the arguments as following:

- **A, WC:** it is obvious that if \mathcal{C}_1 and \mathcal{C}_2 satisfies Axioms A and WC then both $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2$ also satisfies Axioms A, WC.
- **Mon, CRM, CRNM:** Suppose $A = (V, \Pi)$, $A' = (V, \Pi')$, and $S \subset V$ are, respectively, two preference networks and a set considered in Axiom Mon. Then if \mathcal{C}_1 and \mathcal{C}_2 satisfy Mon, we have $S \in \mathcal{C}_i(A') \Rightarrow S \in \mathcal{C}_i(A)$ for $i \in 1, 2$. Thus, if $S \in \mathcal{C}_1(A') \cap \mathcal{C}_2(A')$ then $S \in \mathcal{C}_1(A) \cap \mathcal{C}_2(A)$, and if $S \in \mathcal{C}_1(A') \cup \mathcal{C}_2(A')$ then $S \in \mathcal{C}_1(A) \cup \mathcal{C}_2(A)$. Thus, both $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2$ also satisfy Axioms Mon. We can argue analogously for Axioms CRM and CRNM.

- **Emb**: If both \mathcal{C}_1 and \mathcal{C}_2 satisfy **Emb**, then for any $A = (V, \Pi)$ and any “embedded world” $A' = (V', \Pi')$ such that Π, Π' satisfy the assumption of Axiom **Emb**, we have $\mathcal{C}_i(A') = \mathcal{C}_i(A) \cap 2^{V'}$ for $i \in \{1, 2\}$. So

$$\begin{aligned}\mathcal{C}_1(A') \cap \mathcal{C}_2(A') &= (\mathcal{C}_1(A) \cap 2^{V'}) \cap (\mathcal{C}_2(A) \cap 2^{V'}) = (\mathcal{C}_1(A) \cap \mathcal{C}_2(A)) \cap 2^{V'} \\ \mathcal{C}_1(A') \cup \mathcal{C}_2(A') &= (\mathcal{C}_1(A) \cap 2^{V'}) \cup (\mathcal{C}_2(A) \cap 2^{V'}) = (\mathcal{C}_1(A) \cup \mathcal{C}_2(A)) \cap 2^{V'}.\end{aligned}$$

Thus, both $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2$ also satisfies Axioms **Emb**.

Together, this shows that $\forall \mathcal{C}_1, \mathcal{C}_2 \in \mathcal{C}_B$, $\mathcal{C}_1 \cap \mathcal{C}_2 \in \mathcal{C}_B$ and $\mathcal{C}_1 \cup \mathcal{C}_2 \in \mathcal{C}_B$. Thus, $\mathcal{T}_B = (\mathcal{C}_B, \cup, \cap, \mathcal{C}_{clique}, \mathcal{C}_{all})$ is a lattice with \mathcal{C}_{all} as the lattice’s top and \mathcal{C}_{clique} as the lattice’s bottom (where the former follows from the fact that $\mathcal{C}_{all}(A)$ satisfies Axioms **A**, **Mon**, **CRM**, **CRNM**, **WC** and **Emb**, while the latter follows from Proposition 5).

- **GS, SA**: Assume $\mathcal{C}_1 \in \mathcal{C}$ and $\mathcal{C}_2 \in \mathcal{C}$ satisfy Axioms **GS** and **SA**. We can then argue as for Axiom **Mon** above to show that both $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2$ satisfies Axioms **GS**, **SA**.

Thus, $\mathcal{T} = (\mathcal{C}, \cup, \cap)$ is a lattice. By Theorem 5, $\mathcal{C}_{comprehensive}$ is the lattice’s top and \mathcal{C}_{clique} as the lattice’s bottom of \mathcal{T} . \square

Theorem 6 allows us to have a notion of the closure of an arbitrary community rule with respect to these six axioms. In order to define it, we say that a community rule \mathcal{C}_2 contains a rule \mathcal{C}_1 if $\mathcal{C}_1(A) \subset \mathcal{C}_2(A)$ for all preference networks A .

Theorem 7. *Given a community rule \mathcal{C} , there exists a unique smallest community rule, denoted $\bar{\mathcal{C}}$, that contains \mathcal{C} and satisfies all community axioms besides **SA** and **GS**.*

Proof. Consider the set $\hat{\mathcal{C}}$ of all community rules that contain \mathcal{C} and satisfy these six axioms. Note that it is non-empty because \mathcal{C}_{all} is guaranteed to contain \mathcal{C} . Apart from some technical issues to be addressed below, if we take the intersection of all the communities in this set, the resulting rule $\bar{\mathcal{C}}$ will still satisfy all six axioms by the proof of Theorem 6, and thus be the smallest community rule of the set.

The technical issues to which we alluded above stem from the fact that, in general, the set $\hat{\mathcal{C}}$ contains uncountably many community rules. The community rule $\bar{\mathcal{C}}$ is thus defined by an uncountable intersection, while Theorem 6 *a priori* only allows one to argue about countably many intersections. But it turns out that while $\hat{\mathcal{C}}$ is uncountable, when checking the axioms, one never has to consider more than a finite set of rules, allowing one to apply the reasoning from the proof of Theorem 6 to show that $\bar{\mathcal{C}}$ does satisfy all desired axioms.

To make this precise, we recall that a community rule is given by a sequence of functions, $\mathcal{C}_V : (V, \Pi) \mapsto \mathcal{C}_V(V, \Pi) \subset 2^V - \emptyset$, where V runs over the non-empty finite subsets of countable reference set V_0 . Expressing both $\bar{\mathcal{C}}$ and the rules in $\mathcal{C}' \in \hat{\mathcal{C}}$ as sequences, $\mathcal{C} = (\mathcal{C}_V)$ and $\mathcal{C}' = (\mathcal{C}'_V)$, we have

$$\bar{\mathcal{C}}_V((V, \Pi)) = \bigcap_{\mathcal{C}' \in \hat{\mathcal{C}}} \mathcal{C}'_V((V, \Pi)).$$

However, when verifying the six axioms for $\bar{\mathcal{C}}$, we only have to deal with a given finite set V at a time (or, in the case of Axiom **Emb**, all subsets $V' \subset V$ of a finite set V); and for a finite set V , $\bar{\mathcal{C}}_V$ can be expressed as the intersection over a finite subset of $\hat{\mathcal{C}}$, which means when checking the axioms for $\bar{\mathcal{C}}_V$, we can use Theorem 6. \square

The Intersection lemma serves as a bridge between the two lattices from Theorem 6: We can obtain the lattice $\mathcal{T} = (\mathcal{C}, \cup, \cap, \mathcal{C}_{clique}, \mathcal{C}_{comprehensive})$ from the lattice $\mathcal{T}_B = (\mathcal{C}_B, \cup, \cap, \mathcal{C}_{clique}, \mathcal{C}_{all})$ by intersecting the community functions on the lattice points of \mathcal{T}_B with $\mathcal{C}_{GS} \cap \mathcal{C}_{SA}$, followed by merging the lattices points with identical community functions. By moving the intersection up the lattice \mathcal{T}_B , we can define more inclusive community rules that satisfy all eight axioms. For example, by intersecting the lattice top (\mathcal{C}_{all}) of \mathcal{T}_B with $\mathcal{C}_{GS} \cap \mathcal{C}_{SA}$, we obtain the lattice top ($\mathcal{C}_{comprehensive}$) of \mathcal{T} .

Remark 1. *Note that Theorem 7 and the Intersection Lemma give us a reasonable mapping from arbitrary community rules to community rules that satisfy all our axioms. Namely, for a given community rule, \mathcal{C} , first take the unique smallest community rule that contains \mathcal{C} and satisfies all axioms besides SA and GS (as in Theorem 7), then apply the intersection from the Intersection Lemma. The mapping can therefore be formulated as*

$$\mathcal{C} \mapsto \overline{\mathcal{C}} \cap \mathcal{C}_{SA} \cap \mathcal{C}_{GS}.$$

As an example, consider the community rule \mathcal{C}_1 that admits all singletons (i.e., subsets of size 1) as communities and nothing else. Because \mathcal{C}_1 only violates WC of the axioms besides SA, $\overline{\mathcal{C}_1}$ in addition to all singletons also contains all cliques (thanks to the influence of Emb). From this, all communities that don't satisfy SA are removed: i.e., all singletons that do not rank themselves first. As the reader may have already guessed, what remains happens to be the Clique Rule. In other words,

$$\overline{\mathcal{C}_1} \cap \mathcal{C}_{SA} \cap \mathcal{C}_{GS} = \mathcal{C}_{clique}$$

In a small step up the lattice \mathcal{T}_B from the Clique Rule, we consider the following community function.

Rule 4 (Relaxed Clique Rule). *For a non-negative function $g : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$, a non-empty subset $S \subseteq V$ is a community in $A = (V, \Pi)$ if and only if $\forall u, s \in S, \pi_s(u) \in [1 : |S| + g(|S|)]$. We denote this community function by $\mathcal{C}_{clique(g)}$.*

Proposition 15. $\mathcal{C}_{clique(g)} \in \mathcal{C}_B$ and hence $(\mathcal{C}_{clique(g)} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA})$ satisfies all eight axioms.

Proof. The (straightforward) proof is left to the reader. \square

We will show in Section 5 below that $\mathcal{C}_{comprehensive}$ is highly “non-constructive” by proving that the decision problem for determining whether a subset in a preference network satisfies Axiom Self-Approval or Group Stability is coNP-complete. On the other hand, we will see that the community rule given by $(\mathcal{C}_{clique(g)} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA})$ can be constructive if g is small, see Proposition 17 in Section 5 below.

As g varies from 0 to ∞ , the community function $\mathcal{C}_{clique(g)}$ moves up the lattice \mathcal{T}_B from \mathcal{C}_{clique} to \mathcal{C}_{all} . The intersection with $\mathcal{C}_{SA} \cap \mathcal{C}_{GS}$ provides us a “vertical” glimpse of the taxonomy lattice \mathcal{T} . In particular, as the community rules along this vertical path become more inclusive (when g increases), they become less constructive for community identification. An alternative “vertical” glimpse can be gained by following “harmonious-path” in the lattice \mathcal{T}_B for community rules formulated by pairwise comparisons.

Rule 5 (Harmonious Path). *For $\lambda \in [0 : 1]$, a non-empty subset S is a λ -harmonious community in $A = (V, \Pi)$ if $\forall u \in S, v \in V - S$, at least λ -fraction of $\{\pi_s : s \in S\}$ prefer u over v . We denote this community function by \mathcal{H}_λ .*

Using the similar argument as in Theorem 4, we can prove

Proposition 16. *For all $\lambda \in [0 : 1]$, $\mathcal{H}_\lambda \in \mathcal{C}_B$. Thus, $\mathcal{H}_\lambda \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA}$ satisfies all eight axioms, $\forall \lambda \in [0 : 1]$. Further, for $\lambda \in (1/2 : 1]$, \mathcal{H}_λ satisfies Axiom SA, and therefore all axioms but Axiom GS.*

Proof. The (easy) proof is left to the reader. \square

Therefore, as λ varies from 1 to 0, the community function \mathcal{H}_λ moves up the lattice \mathcal{T}_B from $\mathcal{H}_1 = \mathcal{C}_{clique}$ to $\mathcal{H}_0 = \mathcal{C}_{all}$, and so does its non-constructiveness, see Proposition 18 in Section 5.

5 Complexity of Community Rules

5.1 Complexity of determining Group Stability and Self Approval

In this section, we demonstrate that $\mathcal{C}_{comprehensive}$ is highly “non-constructive” by showing that the decision problem for determining whether a subset in a preference network satisfies Axiom Self-Approval or Group Stability is coNP-complete. Our reduction also provides examples of preference networks derived from 3-SAT instances.

Theorem 8. *It is coNP-complete to determine whether a subset $S \subset V$ is self-approving in a given preference network $A = (V, \Pi)$.*

Before starting the proof, we introduce a notation which we will use throughout this section. Given a preference profile (V, Π) and a non-empty set $S \subset V$, we say that a set $G' \subset V - S$ is a *witness that S is not self-approving*, if S lexicographically prefers G' to S , and we say that a pair $(G, G') \subset S \times (V - S)$ is a *witness that S is not group-stable* if $S - G$ lexicographically prefers G' to G . Finally, we say that $G \subset S$ *threatens the stability of S* if there exists a $G' \subset V - S$ such that $S - G$ lexicographically prefers G' to G .

Proof. We reduce 3-SAT to this decision problem: Suppose $\mathbf{c} = (c_1, \dots, c_m)$ is a 3-SAT instance with Boolean variables $\mathbf{x} = (x_1, \dots, x_n)$ (i.e., $c_j = \{u_j, v_j, w_j\} \subset \cup_{i=1}^n \{x_i, \bar{x}_i\}$). We define a preference network as follows:

- $V = A \cup B \cup D \cup X$ has $m + n + m + 2n$ members, where $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_n\}$, $D = \{d_1, \dots, d_m\}$, and $X = \{x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n\}$. The distinguished subset will be $S = A \cup B$, and for convenience we will denote its complement as $U = D \cup X$.
- Since we will focus on subset S , here we only define the preferences of members in S . The preferences of U can be chosen arbitrarily.
 - Member b_i has preference $D \succ A \succ \{x_i, \bar{x}_i\} \succ \{b_i\} \succ X - \{x_i, \bar{x}_i\} \succ B - \{b_i\}$, where preferences between elements of each set can be chosen arbitrarily.
 - Member a_j has preference $c_j \succ \{a_j\} \succ D \cup X - c_j \succ B \cup A - \{a_j\}$, where again preferences between elements of each set are arbitrary.

Intuitively, members of A are used to enforce clause consistency (i.e., make sure each clause is satisfied) and members of B are used to enforce variable consistency (no variable to both true and false at the same time). Subsets of X naturally constitute an assignment of the variables, and D provides necessary padding in order to apply **Self-Approval**.

We now show that S is not **self-approving** if and only if the 3-SAT instance is satisfiable.

In one direction, suppose $Y = \{y_1, \dots, y_n\}$ where $y_i \in \{x_1, \bar{x}_i\}$ is a satisfying assignment for the 3-SAT instance. Let $G' = Y \cup D$. Now consider the bijection, f , where $f(a_j) = d_j$ and $f(b_i) = y_i$. It is not hard to see that for all $s \in S$ and all i , $f(s) \succ_{\pi_{b_i}} s$. All that is left is to find similar bijections for each a_j . First, note that for a_j all bijections f_j trivially satisfy $f_j(s) \succ_{\pi_{a_j}} s$ where $s \in B \cup A - \{a_j\}$, since this set is ranked at the bottom of π_{a_j} . Therefore it is sufficient to show that there exists an element of G' that a_j prefers to itself. This happens so long as one of the literals from its clause is in G' , which must be true by the fact that Y is a satisfying assignment.

In the other direction, suppose $G' \subset U = D \cup X$ is a witness that S is not self-approving. We note the following:

- $D \subset G'$ otherwise any b_i will have a member of A that cannot be mapped to a more preferred member of G' .
- Let $Y = X \cap G'$. Then $|Y| = n$ by the above fact and the fact that $|G'| = n + m$.
- $\{x_i, \bar{x}_i\} \cap G' \neq \emptyset$ by b_i 's preference, and by the pigeonhole principle the literals of Y are consistent (i.e. $\{x_i, \bar{x}_i\} \not\subseteq Y$).
- $c_j \cap Y \neq \emptyset$ by a_j 's preferences.

Therefore the variable assignment implied by Y is a satisfying assignment for the 3-SAT instance. \square

The following “padding” lemma allows us to reduce various complexity results concerning community axioms to Theorem 8.

Lemma 2. *Let $\emptyset \neq S \subset V \subset V'$ be such that the size of $\tilde{S} = V' - V$ is at least $|S|$, and let $S' = S \cup \tilde{S}$. Then each preference profile Π on V can be mapped onto a preference profile Π' on V' such that*

$$(i) \ S' \in \mathcal{C}_{GS}(V', \Pi') \cap \mathcal{C}_{SA}(V', \Pi') \Leftrightarrow S' \in \mathcal{C}_{GS}(V', \Pi').$$

$$(ii) \ S' \in \mathcal{C}_{GS}(V', \Pi') \Leftrightarrow S \in \mathcal{C}_{SA}(V, \Pi).$$

Proof. Since $|\tilde{S}| \geq |S|$, we can find a surjective map $g : \tilde{S} \rightarrow S$. Given such a map, define Π' arbitrarily, except for the following two constraints:

- If $s \in S$, then π'_s ranks all of $S' = \tilde{S} \cup S$ before anyone in $V - S' = V' - S'$;
- If $\tilde{s} \in \tilde{S}$, then $\pi'_{\tilde{s}}$ ranks all of \tilde{S} first, and then gives the rank $\pi'_{\tilde{s}}(v) = |\tilde{S}| + \pi_{g(\tilde{s})}(v)$ to every $v \in V - V' = V - \tilde{S}$.

Since every $s \in S \subset S'$ ranks all of S' before $V' - S'$, no subset $G' \subset V' - S'$ can be lexicographically preferred by π'_s to a subset of S' . As a consequence, S' is trivially self-improving with respect to Π' , proving statement (i).

Furthermore, G cannot threaten the stability of S' if $G \subset S'$ is such that $(S' - G) \cap S \neq \emptyset$. If $G \subset S'$ threatens the stability of S' , we therefore must have that $G \supset S$. On the other hand, if $G \supsetneq S$, then G contains an element $\tilde{s} \in \tilde{S}$ which means that no set G' can be lexicographically preferred to G , since all elements of S' prefer all of \tilde{S} to anyone in $V' - S'$.

Thus G can only threaten the stability of S' if $G = S$. In other words, $S \notin \mathcal{C}_{GS}(\Pi')$ if and only if there exists $G' \subset V' - S'$ such that for all $\tilde{s} \in \tilde{S} = S' - G$, G' is lexicographically preferred to S with respect to $\pi'_s = \pi_{g(\tilde{s})}$. Since by assumption, the image of \tilde{S} under g is all of S , this is equivalent to the statement that for all $s \in S$, G' is lexicographically preferred to S with respect to π_s , which is the condition that G' is a witness to $S \notin \mathcal{C}_{SA}(\Pi)$, proving statement (ii). \square

Given this lemma, the next two theorems are immediate corollaries to Theorem 8.

Theorem 9. *It is coNP-complete to determine whether a subset $S \subset V$ is group-stable in a given preference network $A = (V, \Pi)$.*

Theorem 10. *It is coNP-complete to determine whether a subset $S \subset V$ is a member of $\mathcal{C}_{comprehensive} = \mathcal{C}_{GS} \cap \mathcal{C}_{SA}$ for a given preference network $A = (V, \Pi)$.*

5.2 Complexity of the rules $\mathcal{C}_{Clique(g)}$ and \mathcal{H}_λ

We now prove although testing membership for $\mathcal{C}_{comprehensive} = \mathcal{C}_{GS} \cap \mathcal{C}_{SA}$ is co-NP complete, the community rule given by $(\mathcal{C}_{Clique(g)} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA})$ can be constructive if g is small.

Proposition 17. *Given a preference network $A = (V, \Pi)$ and a subset $S \subseteq V$, then we can determine in $O(2^g |S|^{g+3})$ time whether or not $S \in (\mathcal{C}_{Clique(g)} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA})(A)$. Particularly, if $g = \Theta(1)$, then this decision problem is in P. However, the decision problem is co-NP complete for $g = |S|^\delta$ for any constant $\delta \in (0, 1]$.*

Proof. It takes time $O(|S|^2)$ to check whether $S \in \mathcal{C}_{Clique(g)}$.

Next we show that it takes time $O(|S|^3 2^g)$ to check if $S \in \mathcal{C}_{SA}(A)$. Indeed, suppose $G' \subseteq V - S$ is a witness that $S \notin \mathcal{C}_{SA}(A)$. We claim that this implies that $G' \subset \pi_s^{-1}([1 : |S| + g]) \forall s \in S$. Suppose this is not true for some $s \in S$. Then $\exists v \in G'$ such that $\pi_s(v) > |S| + g$, which in turn implies that $v \prec_{\pi_s} u \forall u \in G$ as $\pi_s(u) \in [1 : |S| + g] \forall u \in G$. Thus there exists no bijection $f_s : S \rightarrow G'$ with the property $f_s^{-1}(v) \prec_{\pi_s} v$, contradicting the assumption that $G' \subseteq V - S$ is a witness that $S \notin \mathcal{C}_{GS}(A)$. We can thus identify the set of all witnesses as follows: (1) Choose $s \in S$, and let $T_s = \pi_s^{-1}([1 : |S| + g]) - S$. (2) Choose a subset $G' \subseteq T_s$. (3) Test if G' is a witness that $S \notin \mathcal{C}_{SA}(A)$. First note that we are dealing with at most $|S| 2^g$ subsets. By Proposition 1, we can conduct the test of Step 3 performing $|S|$ integer sorting. Thus, the total complexity for Steps 1-3 is $O(|S|^3 2^g)$.

We can similarly test for group stability for $S \in \mathcal{C}_{Clique(g)}$. Suppose (G, G') is a witness that $S \notin \mathcal{C}_{GS}(A)$. Then, it must be the case that $G' \subset \pi_s^{-1}([1 : |S| + g]) \forall s \in S - G$. Suppose this is not true for some $s \in S - G$. Then there must be a $v \in G'$ such that $v \prec_{\pi_s} u, \forall u \in G$ as $u \in \pi_s^{-1}([1 : |S| + g])$, which implies that there exists no bijection $f_s : G \rightarrow G'$ with the

property $f^{-1}(v) \prec_{\pi_s} v$, contradicting the assumption that $(G \subset S, G' \subseteq V - S)$ is a witness that $S \notin \mathcal{C}_{GS}(A)$.

We say G' is a *potential witness* to $S \notin \mathcal{C}_{GS}(A)$ if there exists $G \subset S$, such that (G, G') is a witness to $S \notin \mathcal{C}_{GS}(A)$. We can identify the set of all potential witnesses as follows: (1) Choose $s \in S$, and let $T_s = \pi_s^{-1}([1 : |S| + g]) - S$. (2) Choose a subset of $G' \subseteq T_s$. (3) Test if G' is a potential witness. Again, we are dealing with at most $|S|2^g$ subsets. As there are at most $|S|^{|G'|} \leq |S|^g$ candidates G to test for (using Proposition 1), Steps 1-3 takes at most $O(2^g |S|^{g+3})$ time.

To show that for large g the problem of determining whether or not a set lies in $\mathcal{C}_{Clique(g)} \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA}$ is in co-NP, we reduce the problem to the one of determining whether for a given preference network $A = (V, \Pi)$, a set $S \subset V$ is a member of $(\mathcal{C}_{GS} \cap \mathcal{C}_{SA})(A)$. To define the reduction, we enlarge both V and S by a large, disjoint set \tilde{S} : $V = V' \cup \tilde{S}$, $S' = S \cup \tilde{S}$, where \tilde{S} is chosen large enough to guarantee that $g(|\tilde{S}|) \geq |V|$, implying in particular that $|S'| + g(|S'|) \geq |S'| + |V| \geq |\tilde{S}| + |V| = |V'|$. Due to this fact, we have that $S' \in \mathcal{C}_{Clique(g)}(V', \Pi')$ for all preference profiles Π' on V' . The statement now follows with the help of Lemma 2 and Theorem 8. \square

Our final proposition in this subsection concerns the complexity of determining whether a set S lies in the class \mathcal{H}_λ .

Proposition 18. *Given a preference network $A = (V, \Pi)$ and a subset $S \subseteq V$, we can determine in polynomial time whether $S \in (\mathcal{H}_\lambda \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA})(A)$ if $(1 - \lambda)|S| < 2$, while it is co-NP complete to answer this question if $(1 - \lambda)|S| \geq 16$.*

Proof. We start with the proof of the positive statement. To this end, we first note that it takes $(|V| - |S|)|S|^2 = O(|V|^3)$ comparisons to check whether $S \in \mathcal{H}_\lambda$.

Next we show that if $S \in \mathcal{H}_\lambda$, then the only groups $G \subset S$ that can threaten the stability of S are those for which

$$|S - G| \leq 2\lfloor(1 - \lambda)|S|\rfloor - 1.$$

Indeed, assume that (G, G') is a witness for $S \notin \mathcal{C}_{GS}(\Pi)$, and let $g = |G'|$. The assumption that $S \in \mathcal{H}_\lambda$ then implies that for all $(u, v) \in G \times G' \subset S \times (V - S)$, there are at most

$$m = |S| - \lceil \lambda |S| \rceil = \lfloor (1 - \lambda)|S| \rfloor$$

elements $s \in S - G \subset S$ such that $v \succ_{\pi_s} u$. Thus the sum over all triples $(u, v, s) \in G \times G' \times (S - G)$ obeying this condition can be at most $g^2 m$. On the other hand, if s lexicographically prefers G' over G , the number of pairs $(u, v) \in G \times G'$ obeying the above condition is at least $\frac{g(g+1)}{2}$, given a lower bound of $|S - G| \frac{g(g+1)}{2}$ on the above number of triples. This proves that $|S - G| \leq \frac{2g}{g+1} m$, and hence $|S - G| \leq 2m - 1$, where in the last step we used that both $|S - G|$ and m are integers.

Thus for $(1 - \lambda)|S| < 2$, we may assume that $G - S$ has size 1 (the case $G - S = \emptyset$ is trivial), which shows that there are at most $|S|$ possible choices for G . Given G , we then only have to check whether a potential $G' \subset V - S$ is lexicographically preferred to G by a single linear order π_s , where s is the single element of $S - G$. Using Proposition 1, the existence of such a G' can be checked by greedily choosing the first $|G|$ elements of $V - S$ with respect to π_s . If this set is lexicographically preferred to G , we know that $S \notin \mathcal{C}_{GS}$, and if for all G considered in the first

step, the greedily found G' is not lexicographically preferred to G , $S \in \mathcal{C}_{GS}$. Since all $S \in \mathcal{H}_\lambda$ are self-approving when $\lambda > 1/2$, this completes the proof of the positive statement.

To prove the negative statement, we use that it is NP-complete to determine whether in a formula consisting of 3-clauses, every clause is satisfied by exactly one literal in the clause, and that this problem stays NP complete if we restrict ourselves to the case where each variable appears in exactly 3 clauses (cubic 1-in-3 SAT) [12]. Note that this means that we can partition the set of clauses into $k = 7$ classes such that the clauses in each class don't share any variables (to see this, consider the graph obtained by joining two clauses whenever they share a variable; this graph has maximal degree at most 6, and hence can be colored by 7 colors, given the desired partition).

Thus consider n boolean variables $\{x_1, \dots, x_n\}$ and k sets of 3-in-1 clauses C_i such that the clauses in each C_i have no common variables. We define X as the set of literals, $X = \{x_1, \bar{x}_1, \dots, x_n, \bar{x}_n\}$, and choose two additional sets Y and T , of size n and $2k + 2$, respectively. It will be convenient to label the elements of Y as y_1, \dots, y_n , and the elements of T as $1, \dots, 2k + 2$. Set

$$S = Y \cup T \quad \text{and} \quad V = S \cup X,$$

and choose $\Pi' \in L(V)^V$ of the form

- If $s \in Y$, π'_s ranks all of T first, followed by everyone in Y , followed by everyone in X
- If $s \in T$, π'_s ranks all of T first, then ranks $V - T$ according to a yet to be determined $\pi_s \in L(Y \cup X)$.
- If $v \in V - S$, π'_v is arbitrary.

With this ranking, everyone in Y ranks all of S above all of V , showing that $S \in \mathcal{H}_\lambda((V, \Pi'))$ as long as $|S - Y| \leq (1 - \gamma)|S|$, i.e., as long as $(1 - \gamma)|S| \geq 2(k + 1) = c_2$. It also shows that S is self-approving, since everyone in S prefers all of T to all of $V - S$, which does not allow for a subset $G' \subset V - S$ such that G' is lexicographically preferred to S by everyone in S . By the same reasoning we also see that S is group-stable against any subgroup G which has a non-zero intersection with T . Finally, S is also stable against any subgroup G such that $Y \setminus G \neq \emptyset$, since for such a subset, $S - G$ contains an element $s \in Y$ which prefers everyone in S to everyone outside S .

Thus the only subgroup G against which S could be unstable is the set $G = Y$, i.e., $S \notin (\mathcal{H}_\lambda \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA})(V, \Pi')$ if and only if there exists a subset $G' \subset X$ such that G' is lexicographically preferred to $G = Y$ by everyone in T . We now show that by defining Π appropriately, such a G' exists if and only if the the 1-in-3 SAT problem given by C_1, \dots, C_k has a satisfying assignment.

We first define π_1 and π_2 :

$$\begin{aligned} \pi_1 &= [x_1, \bar{x}_1, y_1, \dots, x_n, \bar{x}_n, y_n] \\ \pi_2 &= [x_n, \bar{x}_n, y_n, \dots, x_1, \bar{x}_1, y_1]. \end{aligned}$$

Clearly, G' is lexicographically preferred to G by both π_1 and π_2 if G' contains exactly one of x_i and \bar{x}_i for each i . On the other hand, if G' is lexicographically preferred to G by π_1 , then by Proposition 1, G' must contain at least one of x_1 and \bar{x}_1 , and if it is lexicographically preferred to G by π_2 , it can contain at most one of x_1 and \bar{x}_1 . Continuing by induction, we see that G'

is lexicographically preferred to G by both π_1 and π_2 if and only if G' contains exactly one of x_i, \bar{x}_i for all i , i.e., if G' corresponds to a truth assignment for the variables x_1, \dots, x_n .

In a similar way, if C_i consists of the clauses $\{z_1, z_2, z_3\}, \{z_4, z_5, z_6\}, \dots, \{z_{3\ell-2}, z_{3\ell-1}, z_\ell\} \subset X$, we define

$$\begin{aligned}\pi_{2i+1} &= [z_1, z_2, z_3, y_1, z_4, z_5, z_6, y_2, \dots, z_{3\ell-2}, z_{3\ell-1}, z_\ell, y_\ell, Q] \\ \pi_{2i+2} &= [Q, z_{3\ell-2}, z_{3\ell-1}, z_\ell, y_\ell, \dots, z_4, z_5, z_6, y_2, z_1, z_2, z_3, y_1],\end{aligned}$$

where Q ranks everyone in $X - \{z_1, \dots, z_{3\ell}\}$ before the remaining elements $y_{\ell+1}, \dots, y_n \in Y$. Now the first ranking enforces that at least one literal of the clause $\{z_1, z_2, z_3\}$ is chosen, while the last enforces that there is at most one such literal. Combining these two and continuing by induction, we see that G' is lexicographically preferred to G by both π_{2i+1} and π_{2i+2} if and only if exactly one literal of each clause in C_i is chosen.

Putting everything together, we see that the 3-in-1 SAT problem has a satisfying assignment if and only if $S \notin (\mathcal{H}_\lambda \cap \mathcal{C}_{GS} \cap \mathcal{C}_{SA})(V, \Pi')$. \square

5.3 Number of Potential Communities

Proposition 19. *Assume that $n \geq 8$. There exists a preference network $A = (V, \Pi)$ such that $\mathcal{C}_{comprehensive}(A) \geq 2^{n/2}$.*

Proof. The preference profile, $\Pi_{H\&S}$, that is about to be described has been dubbed the “hero and sidekick” example as will soon become clear. Consider a world composed of $n/2$ hero-sidekick duos. Each member of a hero-sidekick duo first prefers the hero of that duo then the sidekick of the duo, then all other heroes, followed lastly by all other sidekicks (in some fixed but arbitrary order). Now consider a subset, S , that is composed of all heroes and an arbitrary set of sidekicks. Note that because there are $2^{n/2}$ different sets of sidekicks, it is sufficient to show that S is a community in $\mathcal{C}_{comprehensive}([n], \Pi_{H\&S})$.

First, note that S clearly satisfies SA.

To show that S satisfies GS, consider two sets $G \subset S$ and $G' \subset V - S$ of equal size. We first note that it will be enough to consider the case where $(S - G) \times G$ contains no hero-sidekick pair (u, v) , since otherwise u would prefer v over everyone else, in particular over everyone in G' . Applying this to the sidekicks in G , we conclude that G must contain at least as many heroes as sidekicks. On the other hand, G' can't be lexicographically preferred to G if G contains at least two heroes, showing that only two cases are possible: G consisting of a hero-sidekick pair, or G made up of just a single hero. But neither one leads to a counter example if $|S - G| > |G| = |G'|$, since then we can find an $s \in S - G$ which is not the partner of any sidekick in G' , which means that s prefers the hero in G to everyone in G' . Since S contains all heroes by assumption, we see that S is group stable as soon as $n \geq 8$. \square

6 Stability of Communities

In this section, we consider several stability measures and their impact on community structures. In particular, we focus on preference perturbations in Section 6.1 and the concept of stable fixed points of an aggregation function in Section 6.2. In both subsections, we will use B³CT self-determined communities as our main examples to illustrate these measures. In Section 6.3, we will study the structure of stable harmonious communities.

6.1 Community Stability with Respect to Preference Perturbations

We first study the structure of self-determined communities that remain self-determined even after a certain degree of changes in their members' preferences.

Definition 10 (Preference Perturbations). *Let $\emptyset \neq S \subseteq V$, and let Π, Π' be two preference profiles over V . For $0 \leq \delta \leq 1$, we say Π' is a δ -perturbation of Π with respect to S if*

$$\max_{v \in V} |\{i \in S : \pi_i(v) \neq \pi'_i(v)\}| \leq \delta |S|.$$

Given any community rule \mathcal{C} and a preference network $A = (V, \Pi)$, we say that a community $S \in \mathcal{C}(A)$ is stable under δ -perturbations if $S \in \mathcal{C}((V, \Pi'))$ for all Π' that are δ -perturbation of Π with respect to S .

In other words, a preference profile is a δ -perturbation of another profile if, for each $v \in V$, at most a δ -fraction of the members of S changed their preference of v . Recalling Definition 6, we now state our first stability result for B^3CT self-determined communities.

Proposition 20. *For any preference network $A = (V, \Pi)$, if $S \subset V$ is a B^3CT self-determined community that is stable under δ -perturbations, then $\exists \alpha > \delta$ such that S is an $(\alpha, \alpha - \delta)$ - B^3CT community. Conversely, if $S \subset V$ is an (α, β) - B^3CT self-determined community, then it is stable under $(\alpha - \beta)/2$ -perturbations.*

Proof. Let $u^* = \operatorname{argmin}\{\phi_S^\Pi(u) : u \in S\}$, and let $\alpha = \phi_S^\Pi(u^*)/|S|$. We now prove that the condition of the proposition implies $\alpha > \delta$. Suppose this is not true. Letting $T = \{s \in S : \pi_s(u^*) \leq |S|\}$, we have $|T| = \alpha |S| \leq \delta |S|$. Now consider a preference profile Π' such that for $s \in T$, π'_s shifts the ranking of u^* to n while maintaining the relative rankings of all other elements in π_s , and $\pi'_v = \pi_v \forall v \notin T$. Then, as the ranking of u^* is more than $|S|$ in every π'_s for $s \in S$, we conclude that S is not a B^3CT self-determined community in (V, Π') , contradicting the assumption that S is stable under δ -perturbations. Now let $v^* = \operatorname{argmax}\{\phi_S^\Pi(v) : v \in V - S\}$, and let $\beta = \phi_S^\Pi(v^*)/|S|$. We can similarly show that if S is stable under δ -perturbations, then $\beta < \alpha - \delta$.

The second direction of the proposition is straightforward. \square

Thus, the main result of [3] can be restated as: there are at most $n^{O(1/\delta)}$ B^3CT communities that are stable under δ -perturbations. We further refine the stability studies of community functions by introducing the notion of membership-preserving perturbation:

Definition 11. *Let (V, Π) be a preference network, and let $\emptyset \neq S \subset V$. A preference profile Π' on V is a membership-preserving perturbation of Π with respect to S if $\forall s \in S, \pi_s(S) = \pi'_s(S)$.*

Note that the preference profile Π' considered in both Axiom CRNM and CRM are special cases of membership-preserving perturbations; in Axiom CRNM, Π_S and Π'_S agree on S (i.e., for all $s, u \in S, \pi_s(u) = \pi'_s(u)$, implying in particular $\pi_s(S) = \pi'_s(S)$), and in Axiom CRM, Π_S and Π'_S agree on $V - S$ (i.e., for all $s \in S, v \in V - S, \pi_s(v) = \pi'_s(v)$, implying $\pi_s(V - S) = \pi'_s(V - S)$) and hence also $\pi_s(S) = \pi'_s(S)$.

Theorem 11. *For any preference network $A = (V, \Pi)$, the number of B^3CT communities that are stable under membership-preserving, δ -perturbations of Π is polynomial in $n^{O(1/\delta)}$.*

Proof. It will be sufficient to show that if a B³CT community S is stable under membership-preserving, δ -perturbations of Π , then either

1. S is an $(\alpha, \alpha - \delta)$ -B³CT community for some $\alpha > \delta$, or
2. $\exists s \in S$ such that $\pi_s(S) = [1 : |S|]$.

Indeed, in the first case, there are at most $n^{O(1/\delta)}$ many $(\alpha, \alpha - \delta)$ -B³CT communities by [3], and in the second case, we have that all communities are of the form $S = \pi_v^{-1}([1 : k])$ for some $s \in V$ and $k \in [n]$, showing that there are at most n^2 such communities.

To establish the above statement, letting $u^* = \operatorname{argmin}\{\phi_S^\Pi(u) : u \in S\}$ and $\alpha = \phi_S^\Pi(u^*)/|S|$, we now prove that if S is not an $(\alpha, \alpha - \delta)$ -B³CT community, then there must be $s \in S$, $\pi_s[1 : |S|] = S$. The assumption that S is not an $(\alpha, \alpha - \delta)$ -B³CT community implies that $\phi_S^\Pi(v^*)/|S| \geq \alpha - \delta$ where $v^* = \operatorname{argmax}\{\phi_S^\Pi(v) : v \in V - S\}$. Let $T = \{s \in S : \pi_s(v^*) \leq |S|\}$. Then, $|T| = \phi_S^\Pi(v^*) \geq (\alpha - \delta)|S|$. Since S is a B³CT community, we know that $|T| < \alpha|S|$.

Using these conditions, we now define a perturbed preference profile. Key to our construction is the following observation: For each $s \in S - T$, if $\pi_s(S) \neq [1 : |S|]$, then $(V - S) \cap \pi_s[1 : |S|] \neq \emptyset$. Thus, there exists π'_s that agrees with π_s on S and $\pi'_s(v^*) \leq |S|$ – we can simply swap v^* with any element in $(V - S) \cap \pi_s[1 : |S|]$. Thus, either there exists $s \in S - T$ such that $\pi_s(S) = [1 : |S|]$ (which implies Case 2 above), or $\pi_s(S) \neq [1 : |S|], \forall s \in S - T$. The latter implies that we can find a set $\tilde{S} \subset S - T$ of size $\alpha|S| - |T| \leq \delta|S|$ and a membership-preserving, δ -perturbations Π' of Π such that $\pi'_s(v^*) \leq |S|$ for all $s \in T \cup \tilde{S}$, implying that S is not a B³CT community in (V, Π') . This contradicts the assumption that S is stable under membership-preserving, δ -perturbations of Π . \square

6.2 Stable Fixed-Points of Social Choice

We can also strengthen the concept of fixed points in our social choice based community framework. Particularly, we measure the stability of a community defined by a fixed-point rule according to some variation of the following definition.

Definition 12. (δ -STRONG FIXED POINTS) *Let $A = (V, \Pi)$ be a preference network, $F : L(V)^* \rightarrow \overline{L(V)}$ be a preference aggregation function, and $\delta \in [0 : 1]$ be a coherence parameter. Then, $S \in \mathcal{C}_F((V, \Pi))$ is δ -strong if for $\forall T \subseteq S$ such that $|T| \geq (1 - \delta) \cdot |S|$,*

$$u \succ_{F(\Pi_T)} v \quad \forall u \in S, v \in V - S.$$

Our goal is to understand the influence of a preference aggregation function F and the stability parameter δ ($0 \leq \delta \leq 1$) on the structure of the δ -strong F -self-determined communities.

Before discussing this further, we point out some subtleties that arise when applying Definition 12 to general aggregation functions. We illustrate this subtlety using weighted fixed-point rules, and, in particular, by comparing the community rule defined by the B³CT voting function to that defined by the Borda count.

Recall that in Definition 8, for preference networks with n elements, a preference aggregation function is determined by a sequence of weighting vectors $W = (w^1, w^2, \dots)$ where $w^k \in \mathbb{R}^n$, denotes the weights for the aggregation of k preferences. While this weight vector is independent of k for the Borda count, it in general can be different for each k , and indeed does depend

on k for B³CT voting. Concretely, for the Borda count, every voting member gives scores $n, n-1, \dots, 1$ to the members of V , while in B³CT voting, it gives a score of 1 to the first k in her preference list, making her scores dependent on the total number of voters, k . Thus when defining self-determined communities with the Borda count, one does not need to first anticipate the community size before aggregating the preferences of its members, but when defining self-determined communities with B³CT voting, the weight assigned to an element by a preference depends on the size of the subset under consideration.

In this regard, when measuring the stability of a community S , Definition 12 uses the same weighting vector to evaluate $F(\Pi_T)$ and $F(\Pi_S)$ for the Borda count based community rule, while it uses different weighting vectors to evaluate $F(\Pi_T)$ and $F(\Pi_S)$ for the B³CT community rule, and these weighting vectors depend on $|T|$. Thus, the former application of Definition 12 appears more natural than the latter application.

As a result, we will use the following variation of Definition 12 to measure the strength of a B³CT community.

Definition 13 (δ -Strong B³CT Communities). *For each $T \subseteq V$ and $i \in V$, let $\phi_{T,k}^\Pi(i)$ denote the number of votes that member i would receive if each member $s \in T$ were casting a vote for each of its k most preferred members according to its preference π_s .*

For $\delta \in [0 : 1]$, a non-empty set $S \subseteq V$ is a δ -strong B³CT community in $A = (V, \Pi)$ if $\forall u \in S, v \in V - S$ and $T \subseteq S$ such that $|T| \geq (1 - \delta) \cdot |S|$,

$$\phi_{T,|S|}^\Pi(u) > \phi_{T,|S|}^\Pi(v) \quad \forall u \in S, v \in V - S.$$

Proposition 21. *If S is a δ -strong B³CT community of $A = (V, \Pi)$, then $\exists \alpha \geq \delta$ such that S is an $(\alpha, \alpha - \delta)$ -B³CT community.*

Proof. Let $u^* = \operatorname{argmin}\{\phi_S^\Pi(u) : u \in S\}$, and let $\alpha = \phi_S^\Pi(u^*)/|S|$ and let $v^* = \operatorname{argmax}\{\phi_S^\Pi(v) : v \in V - S\}$, and let $\beta = \phi_S^\Pi(v^*)/|S|$. We now prove that $\alpha - \beta > \delta$.

The pair u^* and v^* partitions S into four subsets.

$$\begin{aligned} S_0 &= \{s \in S : (\pi_s(u^*) \notin [1 : |S|]) \text{ and } (\pi_s(v^*) \notin [1 : |S|])\} \\ S_1 &= \{s \in S : (\pi_s(u^*) \notin [1 : |S|]) \text{ and } (\pi_s(v^*) \in [1 : |S|])\} \\ S_2 &= \{s \in S : (\pi_s(u^*) \in [1 : |S|]) \text{ and } (\pi_s(v^*) \notin [1 : |S|])\} \\ S_3 &= \{s \in S : (\pi_s(u^*) \in [1 : |S|]) \text{ and } (\pi_s(v^*) \in [1 : |S|])\} \end{aligned}$$

Then

$$\begin{aligned} |S_0| + |S_1| + |S_2| + |S_3| &= |S|, \\ |S_1| + |S_3| &= \beta \cdot |S|, \\ |S_2| + |S_3| &= \alpha \cdot |S|, \\ |S_0| + 2|S_1| + |S_3| &< (1 - \delta) \cdot |S|, \end{aligned}$$

where the last inequality follows from the assumption that S is a δ -strong B³CT-self-determined community.

To see this, we first note that $|S_1| < |S_2|$ due to the fact that $\beta < \alpha$. Define T to be the union of S_0 , S_1 , S_3 , and \tilde{S}_2 , where $\tilde{S}_2 \subset S_2$ is an arbitrary subset of size $|S_1|$. Assume by contradiction

that $|T| \geq (1 - \delta)|S|$. Since S is a δ -strong B^3CT -self-determined community, this would imply that $\phi_{T,|S|}(u^*) > \phi_{T,|S|}(v^*)$, i.e.

$$0 < \sum_{s \in T} (1_{\pi_s(u^*) \leq |S|} - 1_{\pi_s(v^*) \leq |S|}).$$

But the right hand side is equal to $|\tilde{S}_2| - |S_1| = 0$, leading to a contradiction. Therefore, $|T| < (1 - \delta) \cdot |S|$, as claimed.

Subtracting the fourth of the above equations from the first, we obtain $(|S_0| + |S_1| + |S_2| + |S_3|) - (|S_0| + 2|S_1| + |S_3|) = (|S_2| + |S_3|) - (|S_1| + |S_3|) > |S| - (1 - \delta) \cdot |S| = \delta \cdot |S|$. Thus, by the second and third equation, we have $\alpha \cdot |S| - \beta \cdot |S| > \delta \cdot |S|$. \square

Proposition 22. *For any $\delta \in (0, 1)$, the number of δ -strong B^3CT communities in any preference network is $n^{O(1/\delta)}$.*

Proof. This follows from the main result of [3] and Proposition 21 above. \square

6.3 Stable Harmonious Communities

Applying the stability notions of Sections 6.1 and 6.2, we define two types of stable harmonious communities. Before doing so, we recall the definition of harmonious communities, Definition 5, and the definition of λ -harmonious communities from Rule 5.

Definition 14 (Stable Harmonious Communities). *For $\delta \in [0 : 1/2]$, a non-empty subset S is a δ -stable harmonious community in $A = (V, \Pi)$ if S is $(\delta + 1/2)$ -harmonious, i.e., if $\forall u \in S, v \in V - S$, at least $(1/2 + \delta)$ -fraction of $\{\pi_s : s \in S\}$ prefer u over v . For $\delta \in [0 : 1]$, S is a δ -strong harmonious community in A if $\forall u \in S, v \in V - S$ and $T \subseteq S$ such that $|T| \geq (1 - \delta) \cdot |S|$, the majority of $\{\pi_s : s \in T\}$ prefer u over v .*

Note that a δ -stable harmonious community is not quite the same as a harmonious community stable under δ -perturbations as defined in Section 6.1. Instead, we have that a δ -stable harmonious community is a harmonious community that is stable under any δ' -perturbations as long as $\delta' < \delta/2$, and that conversely, a harmonious community that is stable under δ perturbation is a δ -stable harmonious community. By contrast, the definition of δ -strong harmonious communities maps exactly to the definition given in Section 6.2.

Proposition 23. *If S is a δ -strong harmonious community, then S is a $\delta/2$ -stable harmonious community.*

Proof. For each pair $u \in S, v \in V - S$, let $f(u, v) = |\{s \in S : u \succ_{\pi_s} v\}| - |\{s \in S : u \prec_{\pi_s} v\}|$ be the *preference gap* between u and v with respect to S . Suppose $(u^*, v^*) = \operatorname{argmin}\{f(u, v) : u \in S, v \in V - S\}$. We now show that if S is a δ -strong harmonious community of A , then $f(u^*, v^*) > \delta \cdot |S|$. The pair u^* and v^* partitions S into two subsets. $S_{\succ} = \{s \in S : u^* \succ_{\pi_s} v^*\}$ and $S_{\prec} = \{s \in S : u^* \prec_{\pi_s} v^*\}$. We have $|S_{\succ}| + |S_{\prec}| = |S|$ and $|S_{\succ}| > |S_{\prec}|$. Let T be the union of S_{\prec} and $|S_{\prec}|$ arbitrary members of S_{\succ} . Since members of T are indifferent about u^* and v^* , we have $|T| = 2|S_{\prec}| \leq (1 - \delta) \cdot |S|$. Thus $f(u^*, v^*) = |S_{\succ}| - |S_{\prec}| = (|S_{\succ}| + |S_{\prec}|) - 2|S_{\prec}| \geq |S| - (1 - \delta) \cdot |S| = \delta \cdot |S|$. Thus, $|S_{\succ}| > (1/2 + \delta/2) \cdot |S|$, and at least $(1/2 + \delta/2)$ -fraction of Π_S prefer u^* over v^* . \square

With a simple probabilistic argument, we can bound the number of δ -stable harmonious communities in any preference networks.

Theorem 12. $\forall \delta \leq 1/2$, the number of δ -stable harmonious communities in any preference network is $n^{12 \log n / \delta^2}$.

Proof. Let S be a δ -stable harmonious communities. For any multi-set $T \subseteq S$, we say T identifies S if for all $u \in S$ and $v \in V - S$, the majority of T prefer u to v . Note that such a T determines S once the size of S is set. To see this, note that the condition implies that $u \succ_{F(\Pi_T)} v$ for all $(u, v) \in S \times (V - S)$, which in turn implies that S is of the form $V_1 \cup \dots \cup V_i$ where (V_1, V_2, \dots) are the components of the ordered partition $F(\Pi_T)$, ordered in such a way that $V_1 \succ_{F(\Pi_T)} V_2, \dots$ (see Proposition 10 and its proof). Thus once $F(\Pi_T)$ and the size of S are fixed, S is uniquely determined.

We now show that $\exists T \subset V$ of size $12 \log n / \delta^2$ that identifies S . To this end, we consider a sample $T \subset S$ of $k = 12 \log n / \delta^2$ randomly chosen elements (with replacements). We analyze the probability that T identifies S . Let $T = \{t_1, \dots, t_k\}$, and for each $u \in S$ and $v \in V - S$, let $x_i^{(u,v)} = [u \succ_{\pi_{t_i}} v]$, where $[B]$ denotes the indicator variable of an event B . Then T identifies S iff $\sum_{i=1}^k x_i^{(u,v)} > k/2, \forall u \in S, v \in V - S$. We now focus on a particular (u, v) pair and bound $\Pr \left[\sum_{i=1}^k x_i^{(u,v)} \leq k/2 \right]$. We first note that

$$\mathbf{E} \left[\sum_{i=1}^k x_i^{(u,v)} \right] = \sum_{i=1}^k \mathbf{E} [x_i^{(u,v)}] \geq \left(\frac{1}{2} + \delta \right) \cdot k.$$

By a standard use of the Chernoff-Hoeffding bound

$$\begin{aligned} \Pr \left[\sum_{i=1}^k x_i^{(u,v)} \leq k/2 \right] &\leq \Pr \left[\sum_{i=1}^k x_i^{(u,v)} \leq (1 + 2\delta)^{-1} \mathbf{E} \left[\sum_{i=1}^k x_i^{(u,v)} \right] \right] \\ &\leq \Pr \left[\sum_{i=1}^k x_i^{(u,v)} \leq (1 - \delta) \mathbf{E} \left[\sum_{i=1}^k x_i^{(u,v)} \right] \right] \\ &\leq e^{-\frac{\delta^2}{2}(1/2+\delta)k} \leq e^{-\frac{\delta^2}{4}k} \leq \frac{1}{n^3}, \end{aligned}$$

where we used that $(1 + 2\delta)^{-1} = 1 - 2\delta(1 + 2\delta)^{-1} \leq 1 - \delta$ in the third step.

If T fails to identify S , then there exists $(u \in S, v \in V - S)$ such that $\sum_{i=1}^k x_i^{(u,v)} \leq k/2$. As there are at most $|S||V - S| \leq n^2$ such (u, v) pairs to consider, by the union bound,

$$\begin{aligned} \Pr [T \text{ identifies } S] &\geq 1 - \sum_{u \in S, v \in V - S} \Pr \left[\sum_{i=1}^k x_i^{(u,v)} \leq k/2 \right] \\ &> 1 - 1/n > 0. \end{aligned}$$

Thus, if S is a δ -stable harmonious communities, then there exists a multi-set $T \subset V$ of size $12 \log n / \delta^2$ that identifies S . We can thus enumerate all δ -stable harmonious communities by

enumerating all (T, t) pairs, where T ranges from all multi-subsets of V of size $12 \log n / \delta^2$ and $t \in [1 : n]$ and check if T can identify a set of size t . \square

7 Remarks

While the results of this paper are conceptual and are built on the abstract framework of preference networks, we hope this study is a significant step towards developing a rigorous theory of community formation in social and information networks. In particular, we hope this will be used to inform and choose among other approaches to community identification which have been developed. Below we discuss a few short-term research directions that may help to expand our understanding in order to make more effective connection with community identification in networks that arise in practice.

Preferences Models

We have based our community formation theory on the *ordinal* concept of utilities used in social choice and modern economic theory [2]. The resulting preference network framework, like that in the classic studies of voting [2] and stable marriage [7], enables our axiomatic approach to focus on the conceptual question of network communities rather than the more practical question of community formation in an observed social network. To better connect with the real-world community identification problem, we need to loosen both the assumption of strict ranking and the assumption of complete preference information.

With simple modifications to our axioms, we can extend our entire theory to a preference network $A = (V, \Pi)$ that allows *indifferences*, i.e., Π is given by n ordered partitions $\{\pi_1, \dots, \pi_n\} : \pi_i \in \overline{L(V)}$. This extension enables us to partially expand our results to affinity networks. Recall an affinity network $A = (V, W)$ is given by n vectors $W = \{w_1, \dots, w_n\}$, where w_i is an n -place non-negative vectors. We can extract an ordinal preference $\pi_i \in \overline{L(V)}$ from the cardinal affinities by sorting entries in w_i – elements with the same weight are assigned to the same partition.

Although this conversion may lose some valuable affinity information encoded in the numerical values, it offers a path for us to apply our community theory – even in its current form – to network analysis. For example, as suggested in [3], given a social network $G = (V, E)$, we can first define an affinity network $A = (V, W)$ where w_i is the personalized PageRank vector of vertex i , and then obtain an preference network (V, Π) where $\pi \in \overline{L(V)}$ ranks vertices in V by i 's PageRank contributions [1] to them.

Theoretically, we would like to extend our work to preference networks with partially ordered preferences as a concrete step to understand community formation in networks with incomplete or incomparable preferences. Like our current study, we believe that the existing literature in social choice – e.g., [16] – will be valuable to our understanding. We expect that an axiomatic community approach to preference networks with partially ordered preferences, together with an axiomatization theory of personalized ranking in a network, may offer us new understanding of how to address the two basic mathematical problems – extension of individual affinities/preferences to community coherence and inference of missing links – for studying communities in a social and information network. As this part of community theory becomes sufficiently well developed, well-designed experiments with real-world social networks will be necessary to further enhance this theoretical framework.

Structures, Algorithms, and Complexity

Our taxonomy theorem provides the basic structure of communities in a preference network, while the coNP-Completeness result illustrates the algorithmic challenges for community identification in addition to community enumeration. On the other hand, our analysis of the harmonious rule and the work of [3] seem to suggest some efficient notion of communities can be defined.

However, it remains an open question if there exists a natural and constructive community rule that simultaneously (i) satisfies all axioms, (ii) allows overlapping communities, and (iii) has stable communities which are polynomial-time samplable and enumerable.

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